Two spanning disjoint paths with required length in generalized hypercubes

Dyi-Rong Duh\textsuperscript{a,\ast}, Yao-Chung Lin\textsuperscript{b}, Cheng-Nan Lai\textsuperscript{c}, Yue-Li Wang\textsuperscript{d}

\textsuperscript{a} Department of Computer Science and Information Engineering, Hwa Hsia Institute of Technology, New Taipei City 23568, Taiwan
\textsuperscript{b} Department of Computer Science and Information Engineering, National Chi Nan University, Puli, Nantou Hsien 54561, Taiwan
\textsuperscript{c} Department of Information Management, National Kaohsiung Marine University, Kaohsiung City 81143, Taiwan
\textsuperscript{d} Department of Information Management, National Taiwan University of Science and Technology, Taipei City 10607, Taiwan

\textbf{Abstract}

Given two pairs \(\langle u, v \rangle\) and \(\langle x, y \rangle\) of vertices of a graph \(G = (V, E)\) and two integers \(l_1\) and \(l_2\) with \(l_1 + l_2 = |V(G)| - 2\), \(G\) is said to be satisfying the 2RP-property if there exist two disjoint paths \(P_1\) and \(P_2\) such that \((1)\) \(P_1\) is a path joining \(u\) to \(v\) with \(l(P_1) = l_1\), \((2)\) \(P_2\) is a path joining \(x\) to \(y\) with \(l(P_2) = l_2\), and \((3)\) \(P_1 \cup P_2\) spans \(G\), where \(l(P)\) denotes the length of path \(P\). In this paper, we show that an \(r\)-dimensional generalized hypercube, denoted by \(G(m_r, m_{r-1}, \ldots, m_1)\), satisfies the 2RP-property except some special conditions, where \(m_i \geq 4\) for all \(1 \leq i \leq r\).

\textcopyright 2013 Elsevier B.V. All rights reserved.

\textbf{1. Introduction}

The topological structure of a multiprocessor system can be modeled by an interconnection network (or a graph), and the interconnection network plays an important role in some issues such as communication performance [3], hardware cost [1], embedding and fault tolerant capabilities [10,16,21] and those are all driven by a mathematical model. Paths are suitable for designing simple algorithms with low communication costs. The path embedding problem is a very important issue for a network and is widely discussed in many researches [2,5,6,8,15,18,19]. Generally, an interconnection network can be modeled by a graph, and edges (respectively, vertices) in the graph represent links (respectively, nodes) in the interconnection network.

An interconnection network is usually represented by an undirected simple graph \(G = (V, E)\), where \(V(G)\) and \(E(G)\) denote the vertex and edge sets of \(G\), respectively. An edge between two vertices \(u\) and \(v\) in \(G\) is denoted by \((u, v)\). For two distinct vertices \(u, v \in V(G)\), \(u\) and \(v\) are adjacent if \((u, v) \in E(G)\). A path in \(G\) is a sequence of edges that connect adjacent vertices. A Hamiltonian path is a path that contains every vertex of \(G\) exactly once. A graph \(G\) is Hamiltonian-connected if every two distinct vertices of \(G\) are connected by a Hamiltonian path. A graph \(G\) is \(k\)-vertex fault-tolerant Hamiltonian-connected (k-Hamiltonian-connected for short) if it remains Hamiltonian-connected after removing no more than \(k\) vertices from \(G\).

The interconnection network considered in this work is the generalized hypercube which has excellent topological properties, such as logarithmic diameter [12], vertex-symmetry [12], edge-symmetry [12], efficient communication [7,12,22] and...
high degree of fault tolerance [21]. Some basic properties of a generalized hypercube, such as diameter, wide diameter, and fault diameter have been determined in [4].

In [13], Lee et al. introduced an interesting property, called 2RP-property, as described below. Let \( l(P) \) denote the length of a path \( P \), i.e., the number of edges which connect adjacent vertices in \( P \). The distance between vertices \( u \) and \( v \) in \( G \), denoted by \( d_G(u, v) \), is the minimum \( l(P) \) for every \( P \) joining \( u \) and \( v \). Given any four distinct vertices \( u, v, x, \) and \( y \) in a graph \( G \), let \( l_1 \) and \( l_2 \) be two integers such that \( l_1 \geq d_G(u, v), l_2 \geq d_G(x, y) \), and \( l_1 + l_2 = |V(G)| - 2 \). Then, there exist two vertex-disjoint (disjoint for short) paths \( P_1 \) and \( P_2 \) such that (1) \( P_1 \) is a path joining \( u \) and \( v \) with \( l(P_1) = l_1 \); (2) \( P_2 \) is a path joining \( x \) and \( y \) with \( l(P_2) = l_2 \), and (3) \( P_1 \cup P_2 \) spans \( G \). Two paths are vertex-disjoint if they do not share any common vertex. The 2RP-property has been studied on various graphs, such as hypercubes [13], augmented cubes [14], arrangement graphs [20] and others [11,17]. Since generalized hypercubes are a generalization of hypercubes, this work further shows that generalized hypercubes satisfy 2RP-property.

The rest of this paper is organized as follows. Section 2 formally introduces the definition and some properties of generalized hypercubes. Section 3 defines the 2RP-property for generalized hypercubes and proves that generalized hypercubes satisfy the 2RP-property. Conclusions are finally drawn in Section 4.

2. Background and notations

This section discusses the structure and some properties of generalized hypercubes. Some notations frequently used in this work are also presented.

2.1. Generalized hypercubes

Generalized hypercubes are a generalization of hypercubes [1] which were proposed for building massively parallel computer systems. An \( r \)-dimensional generalized hypercube, denoted by \( G(m_r, m_{r-1}, \ldots, m_1) \), is of order \( \prod_{i=1}^{r}(m_i) \), where \( r \geq 1 \) and \( m_i \geq 2 \) for all \( 1 \leq i \leq r \). Each vertex in \( G(m_r, m_{r-1}, \ldots, m_1) \) is assigned an \( r \)-digit identifier \( x_i x_{i-1} \ldots x_1 \), where \( x_i = 0, 1, \ldots, m_i - 1 \) for all \( 1 \leq i \leq r \). Two vertices in \( G(m_r, m_{r-1}, \ldots, m_1) \) are adjacent if and only if their identifiers differ at exactly one digit position. Two adjacent vertices whose identifiers differ at position \( i \) are connected by an \( i \)-edge and they are \( i \)-neighbors of each other, where \( 1 \leq i \leq r \). Thus each vertex in \( G(m_r, m_{r-1}, \ldots, m_1) \) has degree \( \sum_{i=1}^{r}(m_i - 1) \).

For a \( G(m_r, m_{r-1}, \ldots, m_1) \), let \( G'[j] \) for \( 0 \leq j \leq m_j - 1 \) stand for the induced subgraph of \( G \) with vertex set \( V(G'[j]) = \{ x_i x_{i-1} \ldots x_1 | 0 \leq x_i \leq m_i - 1 \text{ for } 1 \leq i \leq r \} \). Let \( N' \) denote the order of \( G(m_{r-1}, m_{r-2}, \ldots, m_1) \), namely, \( |V(G'[j])| = N' \) for all \( 0 \leq j \leq m_j - 1 \). Each \( G'[j] \) for \( 0 \leq j \leq m_j - 1 \) can be regarded as a supernode and two supernodes \( G'[i] \) and \( G'[j] \) are adjacent if there exists an edge \((u, v)\) between \( u \in V(G'[i]) \) and \( v \in V(G'[j]) \). Thus, \( G(m_r, m_{r-1}, \ldots, m_1) \) can be regarded as a complete graph of \( m_r \) supernodes.

Let \( x = x_i x_{i-1} \ldots x_1 \) be a vertex in \( G(m_r, m_{r-1}, \ldots, m_1) \) and let \( f^j(x) = x_i \) for \( 1 \leq i \leq r \). Also, let \( x_{i(j)} \) in \( G'[j] \) be the \( r \)-neighbor of \( x \) in \( G'[x_i] \) for \( 0 \leq j \leq m_j - 1 \) and \( j \neq x_i \). Notably, the order of dimensions of \( G(m_r, m_{r-1}, \ldots, m_1) \) can be exchanged without changing its structure. For instance, \( G(4, 3, 2) \) is isomorphic to \( G(2, 3, 4) \). Fig. 1 shows the structure of \( G(4, 3, 2) \). In Fig. 1, if \( x = x_3 x_2 x_1 \) = 001, then \( x \) is in \( G^2[0] \) and its \( r \)-neighbors are \( x(1) = 101, x(2) = 201 \) and \( x(3) = 301 \).

Let \( N_G(m_r, m_{r-1}, \ldots, m_1)(u) \) denote the set of vertices adjacent to vertex \( u \) in \( G \). When then context is clear, \( N_G(m_r, m_{r-1}, \ldots, m_1)(u) \) and \( d_G(m_r, m_{r-1}, \ldots, m_1)(u, v) \) are simply written as \( N_G(u) \) and \( d_G(u, v) \), respectively. For convenience, let \( (u, u_0, u_1, \ldots, u_m, v) \) denote a path joining \( u \) and \( v \), where all the vertices \( u, u_0, u_1, \ldots, u_m, \) and \( v \) are distinct. For brevity, \( (u, u_0, u_1, \ldots, u_m, v) \) is represented by \( (u, P, v) \) if \( P = (u, u_0, u_1, \ldots, u_m, v) \).
2.2. Some properties of generalized hypercube graphs

This section addresses some properties of generalized hypercubes, and proves that it is a 1-Hamiltonian-connected graph.

**Lemma 1.** (See [9].) \( G(m_r, m_{r-1}, \ldots, m_1) \) is Hamiltonian-connected except \( m_i = 2 \) for all \( 1 \leq i \leq r \).

**Lemma 2.** \( G(m_r, m_{r-1}, \ldots, m_1) \) is 1-Hamiltonian-connected, where \( m_i \geq 3 \) for all \( 1 \leq i \leq r \).

**Proof.** We prove this lemma by induction on dimension \( r \). For the basis step, i.e., \( r = 1 \), \( G(m_1) \) is a complete graph of \( m_1 \) vertices and the basis clearly holds. Assume that the lemma holds for any \( G(m_r, m_{r-1}, \ldots, m_1) \) with \( r \leq k \).

Let \( f \) be the faulty vertex in \( G(m_{k+1}, m_k, \ldots, m_1) \), and \( u, v \in V(G(m_{k+1}, m_k, \ldots, m_1)) \setminus \{f\} \) be two distinct vertices. Since \( u \neq v \), there exists a position \( l \) such that \( \sigma^l(u) \neq \sigma^l(v) \), where \( 1 \leq l \leq k+1 \). Without loss of generality, we may assume that \( l = k+1 \), \( u \in V(G^{k+1}[0]) \) and \( v \in V(G^{k+1}[1]) \). We consider the following two cases.

**Case 1.** \( \sigma^{k+1}(f) \notin [\sigma^{k+1}(u), \sigma^{k+1}(v)] \): Without loss of generality, assume that \( \sigma^{k+1}(f) = 2 \), and thus \( f \) is included in \( G^{k+1}[2] \).

**Case 1.1.** \( (k+1) \geq 4 \): Clearly, there exists a vertex, say \( p(0) \), in \( G^{k+1}[0] \) such that \( p(2) \neq f \). Furthermore, there exists a vertex, say \( q(2) \), in \( G^{k+1}[2] \) such that \( q(1) \neq v \). By **Lemma 1**, there exists a Hamiltonian path \( (u, H_1, p(0)) \) in \( G^{k+1}[0] \) and a Hamiltonian path \( (q(1), H_3, v) \) in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus G^{k+1}[0] \setminus G^{k+1}[2] \). By the induction hypothesis, there exists a 1-Hamiltonian path \( (p(2), H_2, q(2)) \) in \( G^{k+1}[2] \) without containing the faulty vertex \( f \). Combining \( H_1, H_2 \) and \( H_3 \), this results in a 1-Hamiltonian path \( (u, H_1, p(0), p(2), H_2, q(2), q(1), H_3, v) \) in \( G(m_{k+1}, m_k, \ldots, m_1) \) without containing the faulty vertex \( f \) (see **Fig. 2**).

**Case 1.2.** \( (k+1) = 3 \): There exists a vertex \( p(2) \) in \( G^{k+1}[2] \setminus \{f\} \) such that \( p(0) \neq u \) and \( q(2) \) is included in \( G^{k+1}[2] \setminus \{f, p(2)\} \). The discussions depend on the locations of \( q(1) \) and \( q(0) \). When \( q(1) \neq v \), the proof is the same as that in **Case 1.1**. When \( q(1) = v \) and \( q(0) \neq u \), by **Lemma 1**, there exists a Hamiltonian path \( (p(1), H, v) \) in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus G^{k+1}[0] \setminus G^{k+1}[2] \). Hence, \( (u, p(0), q(0), q(2), p(2), p(1), H, v) \) is the Hamiltonian path in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus \{f\} \) as shown in **Fig. 3**. When \( q(1) = v \) and \( q(0) = u \), by **Lemma 1**, there exists a Hamiltonian path \( (f(1), H, v) \) in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus G^{k+1}[0] \setminus G^{k+1}[2] \). Hence, \( (u, q(2), p(2), p(0), f(0), f(1), H, v) \) is the 1-Hamiltonian path in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus \{f\} \) as depicted in **Fig. 4**.

**Case 2.** \( \sigma^{k+1}(f) \in [\sigma^{k+1}(u), \sigma^{k+1}(v)] \): Without loss generality, assume that \( \sigma^{k+1}(f) = \sigma^{k+1}(u) \), and then assume that both \( f \) and \( u \) are included in \( G^{k+1}[0] \). There exists a vertex \( p(0) \) in \( G^{k+1}[0] \setminus \{f, u\} \). By **Lemma 1**, there exists a Hamiltonian path \( (p(2), H_1, v) \) in \( G(m_{k+1}, m_k, \ldots, m_1) \setminus G^{k+1}[0] \) because \( \sigma^{k+1}(p(2)) \neq \sigma^{k+1}(v) \). By the induction hypothesis, there exists a 1-Hamiltonian path \( (u, H_2, p(0)) \) in \( G^{k+1}[0] \) in which \( f \) is a faulty vertex. Obviously, \( (u, H_2, p(0), p(2), H_1, v) \) is a 1-Hamiltonian path in \( G(m_{k+1}, m_k, \ldots, m_1) \) with faulty vertex \( f \).

**Lemma 3.** Let \( P \) be a path in \( G(m_r, m_{r-1}, \ldots, m_1) \). If \( l(P) \geq m_r \), then \( P \) contains an \( m \)-edge where \( m \neq r \).

**Proof.** Let \( P = (u_0, u_1, \ldots, u_l) \) be a path in \( G(m_r, m_{r-1}, \ldots, m_1) \), where \( k \geq m_r \), and any two vertices of \( P \) are connected by \( r \)-edges. In other words, \( \sigma^i(u_{i+1}) \neq \sigma^i(u_{j+1}) \) where \( 0 \leq i, j \leq k \) and \( i \neq j \). Thus, there should be \( k+1 > m_r \) distinct vertices. It is a contradiction because there are only \( m_r \) distinct vertices in \( r \) dimension. Therefore, this lemma holds.

![Fig. 2. The Hamiltonian path constructed by Case 1.1.](image-url)
Lemma 4. Let $P$ be a path from $u$ to $v$, and $w$ and $x$ be any two distinct vertices in $G(m_r, m_{r-1}, \ldots, m_1)$, where $r \geq 2$ and $m_i = 4$ for all $1 \leq i \leq r$. If $I(P) = 4' - 3$, then there exist two adjacent vertices $y$ and $z$ in $P$ such that $N_C(w) \cap N_C(x) \neq \emptyset$ and $\{w, x\} \cap \{y, z\} = \emptyset$.

Proof. Since $|N_C(w)| = 3r$, $|V(P) - N_C(w) - \{u, v, w, x\}| \geq (4' - 2) - 3r - 4 \geq 1$. Thus, there exists at least one vertex, say $y$, in $V(P) - N_C(w) - \{u, v, w, x\}$. Because $|N_P(y)| = 2$ and $y \notin N_C(w)$, there exists a vertex $z \in N_P(y)$ such that $z \notin \{w, x\}$. This further implies that $N_C(w) \cap N_C(x) \neq \emptyset$ and $\{w, x\} \cap \{y, z\} = \emptyset$. This completes the proof. $\square$

3. 2RP-property

This section demonstrates that $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, when $m_i \geq 4$ for all $1 \leq i \leq r$. Let $u, v, x$ and $y$ be any four distinct vertices of $G(m_r, m_{r-1}, \ldots, m_1)$, and $l_1$ and $l_2$ be two integers with $l_1 \geq d_G(u, v)$, $l_2 \geq d_G(x, y)$ and $l_1 + l_2 = |V(G(m_r, m_{r-1}, \ldots, m_1))| - 2$. In the rest of this paper, we may assume without loss of generality that $l_1 \leq l_2$ unless stated otherwise. Then, there exist two disjoint paths $P_1$ and $P_2$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = l_1$; (2) $P_2$ is a path joining $x$ to $y$ with $l(P_2) = l_2$, and (3) $P_1 \cup P_2$ spans $G(m_r, m_{r-1}, \ldots, m_1)$ except the following two cases: (a) $l_1 = 2$, $d_G(u, v) = 1$ and $N_C(u) \cap N_C(v) = \{x, y\}$; (b) $l_1 = 2$, $d_G(u, v) = 2$ and $N_C(u) \cap N_C(v) = \{x, y\}$. Note that if $l_1 = 2$, then it requires that there is a path $P_1$ of length 2 from $u$ to $v$. In this case, $P_1$ must pass through a vertex in $N_C(u) \cap N_C(v)$. However, $N_C(u) \cap N_C(v) = \{x, y\}$. This implies that $P_1$ and $P_2$ are not disjoint.

Proposition 1. $G(4, 4)$ satisfies the 2RP-property.

Proof. In $G(4, 4)$, there are exactly 16 vertices. Thus, $1 \leq l_1 \leq \lfloor |V(G(4, 4))| - 2 / 2 \rfloor = 7$. Note that $G(4, 4)$ is the Cartesian product graph of $K_4 \times K_4$, where $K_4$ denotes a complete graph of four vertices. First we consider the case where the Hamming distance between $u$ and $v$ is 1. By the definition of the 2RP-property, at most one of $x$ and $y$ can be in the same $K_4$ with $u$ and $v$; otherwise, $N_C(u) \cap N_C(v) = \{x, y\}$ which is an exception when $l_1 = 2$. Thus, we can find paths $P_1$ of lengths 1 and 2 in the $K_4$ containing $u$ and $v$. Clearly, there exists a $K_4$ in $G(4, 4)$ in which none of $u, v, x$ and $y$ is in it, and we can construct $P_1$ of lengths from 3 to 5 by only using the vertices in this $K_4$ and $u$ and $v$. To construct $P_1$ of lengths 6 and 7, adding one vertex in each remaining $K_4$ to the above selected vertices on constructing a $P_1$ of length 5.
Now we consider the case where the Hamming distance between $u$ and $v$ is 2. This implies that $u$ and $v$ are in two different $K_4$. Clearly, there is a $P_1$ of length 2 from $u$ to $v$. If $x$ and $y$ are in the same $K_4$, then there exists a $K_4$ in which none of $u$, $v$, $x$ and $y$ is in it, and we can construct $P_1$ of lengths from 3 to 5 by using the vertices in this $K_4$. If $x$ and $y$ are in the same $K_4$ containing $u$ (respectively, $v$), then all vertices in the $K_4$ containing $v$ (respectively, $u$) as well as the vertices for constructing $P_1$ of length 5 can be used to construct $P_1$ of lengths 6 and 7. If each $K_4$ contains a vertex in $\{u, v, x, y\}$, then, by using the vertices in those two $K_4$ containing $u$ and $v$, a $P_1$ of lengths from 2 to 7 can be constructed.

By inspection, it is easy to verify that, after removing the vertices in $P_1$ from $G(4, 4)$, there exists a $P_2$ of length $l_2$. This completes the proof. \hfill \Box

Lemma 5. $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies 2RP-property, where $r \geq 2$ and $m_i = 4$ for all $1 \leq i \leq r$.

Proof. This lemma is proved by induction on $r$. By Proposition 1, the basis holds for $r \leq 2$. Suppose the lemma holds for any $G(m_r, m_{r-1}, \ldots, m_1)$ with $r \leq k$. To prove the lemma holds, let $r = k + 1$ and $G$ denote $G(m_{k+1}, m_k, \ldots, m_1)$ for simplicity. Since $l_1 \leq 2$, $l_1 \leq (4^{k+1} - 2)/2 = 2(4^k) - 1$. Let $u$, $v$, $x$ and $y$ be four arbitrary distinct vertices in $G$. Since $x \neq y$, there exists a digit $i$ such that $\sigma^i(x) \neq \sigma^i(y)$, where $1 \leq i \leq k + 1$. Without loss of generality, we may assume that $l = k + 1$ and that $x \in V(G(4^k[1]))$ and $y \in V(G(4^k[1]))$. Five cases need to be considered as follows.

1. $(\sigma^{k+1}(u) \neq \sigma^{k+1}(v)$ and $[\sigma^{k+1}(u), \sigma^{k+1}(v)] \cap [\sigma^{k+1}(x), \sigma^{k+1}(y)] = \emptyset$): Without loss of generality, assume that $\sigma^{k+1}(u) = 2$, $\sigma^{k+1}(v) = 3$, such that $u \in G(4^k[2])$, $v \in G(4^k[3])$. Three subcases are necessary to be considered depending on the values of $d_G(u, v)$ and $l_1$. Note that, based on the number of vertices in $G(4^k[1])$, the length of $l_1$ can be separated to $d_G(u, v) \leq l_1 \leq 4^k - 1$ and $4^k \leq l_1 \leq 2(4^k) - 1$ in these subcases.

2. $(d_G(u, v) \leq l_1 \leq 4^k - 1$ with $d_G(u, v) = 1$): For $l_1 = 1$, we have $P_1 = (u, v)$. There is a vertex $a_{01} \in G(4^k[0]) \setminus \{x, y\}$ such that $a_{(1)} \neq y$. By Lemma 1, there are two Hamiltonian paths $\langle x, H, a_{01} \rangle$ in $G(4^k[0])$ and $\langle a_{11}(1), Q, y \rangle$ in $G(4^k[1])$. Since $l(Q) = 4^k - 1$, $Q$ can be written as $\langle a_{01}, Q_1, b_1(1), c_{11}, Q_2, y \rangle$ for some vertices $b_{1(1)}$ and $c_{11}$ such that $\{b_{1(1)}, c_{11}\} \cap \{u \} = \emptyset$. According to Lemma 2, there is a Hamiltonian path $\langle b_{1(2)}, R, c_{1(2)} \rangle$ in $G(4^k[2]) \setminus \{u\}$, and $R$ can be written as $\langle b_{1(2)}, R_1, d_{1(2)}, e_{1(2)}, R_2, c_{1(2)} \rangle$ for some vertices $d_{1(2)}$ and $e_{1(2)}$. By Lemma 2, there is a Hamiltonian path $\langle d_{1(3)}, S, e_{1(3)} \rangle$ in $G(4^k[3]) \setminus \{v\}$. Hence, $P_2 = \langle x, H, a_{01}, a_{11}, Q_1, b_{1(2)}, b_{1(2)}, R_1, d_{1(2)}, d_{1(3)}, S, e_{1(2)}, e_{1(2)}, R_2, c_{1(2)}, c_{1(2)}, Q_2, y \rangle$, and $P_1$ and $P_2$ are the required paths as shown in Fig. 5.

3. For $l_1 = 2$, only $N_G(u) \cap N_G(v) \neq \emptyset \{x, y\}$ needs to be discussed. Since $N_G(u) \cap N_G(v) = \{u_{00}, u_{11}\}$, we have $\{u_{00}, u_{11}\} \neq \{x, y\}$. Without loss of generality, assume that $u_{00} \neq x$. Hence, $P_1 = (u, u_{00}, v)$. By Lemma 2, there is a Hamiltonian path $\langle x, H', a_{01} \rangle$ in $G(4^k[0]) \setminus \{u_{00}\}$. Hence, $P_2 = \langle x, H', a_{01}, a_{11}, Q_1, b_{1(2)}, b_{1(2)}, R_1, d_{1(2)}, d_{1(3)}, S, e_{1(2)}, e_{1(2)}, R_2, c_{1(2)}, c_{1(2)}, Q_2, y \rangle$, and $P_1$ and $P_2$ are the required paths as shown in Fig. 6.

4. For $3 \leq l_1 \leq 4^k - 1$, there are two vertices $a_{2(2)} \in N_G[\{1\}] \setminus \{x, y\}$ such that $b_{1(1)} \neq y$. By Lemma 1, there are two Hamiltonian paths $H$ in $G(4^k[0])$ and $Q$ in $G(4^k[1])$ such that $H$ joins $x$ and $b_{1(0)}$, and $Q$ joins $b_{1(1)}$ and $y$. Since $l(H) = 4^k - 1$, $H$ can be written as $\langle b_{1(1)}, Q_1, c_{11}, d_{1(1)}, Q_2, y \rangle$ for some vertices $c_{11}$ and $d_{1(1)}$ such that $\{c_{11}, d_{1(1)}\} \cap \{u_{00}, u_{11}\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $R_1$ and $R_2$ such that $(1)$ $R_1$ is a path joining $u$ and $a_{2(2)}$ with $l(R_1) = 1$; (2) $R_2$ is a path joining $c_{1(2)}$ and $d_{1(2)}$ with $l(R_2) = 4^k - 3$, and $(3)$ $R_1 \cup R_2$ spans $G(4^k[2])$. By Lemma 4, $R_2$ can be written as $\langle c_{1(2)}, R_1, R_2, d_{1(2)} \rangle$ for some vertices $e_{1(2)}$ and $f_{1(2)}$ such that $N_G(\{1\}) \cap N_G(\{2\}) = \emptyset$. By the induction hypothesis, there exist two disjoint paths $S_1$ and $S_2$ such that $(1)$ $S_1$ is a path joining $a_{2(2)}$ and $v$ with $l(S_1) = l_1 - 2$; (2) $S_2$ is a path...
joining \(e_3\) and \(f_3\) with \(l(S_2) = 4^k - 2 - l(S_1)\), and (3) \(S_1 \cup S_2\) spans \(G^{k+1}[3]\). Hence, \(P_1 = (u, R_1, a_2, a_3, S_1, v)\) and \(P_2 = (x, H, b_0, b_1, Q_1, c_1, c_2, R_2, e_2, e_3, S, f_3, f_2, R_22, d_2, d_1, Q_2, y)\) are the required paths as shown in Fig. 7.

Case 1.2. \((d_C(u, v) \leq l_1 \leq 4^k - 1 \text{ with } d_C(u, v) \geq 2)\): For \(d_C(u, v) \leq l_1 \leq 4^k - 2\), there is a vertex \(a_0\) in \(G^{k+1}[0]\) such that \(a(1) \neq y\) and \(a(3) \notin [u(3), v]\). By Lemma 1, there are two Hamiltonian paths \(H\) in \(G^{k+1}[0]\) and \(Q\) in \(G^{k+1}[1]\) such that \(H\) joins \(x\) and \(a_0\), and \(Q\) joins \(a(1)\) and \(y\). Since \(l(Q) = 4^k - 1\), \(Q\) can be written as \(a(1), Q_1, b_1, c_1, Q_2, y\) for some vertices \(b_1\) and \(c_1\) such that \([b_1, c_1] \cap [u] = \emptyset\). By Lemma 2, there is a Hamiltonian path \((b_2, R, c_2)\) in \(G^{k+1}[2]\) with \(u\). By Lemma 4, \(R\) can be written as \((b_2, R_1, d_2, e_2, R_2, c_2)\) for some vertices \(d_2\) and \(e_2\) such that \(NG_{G^{k+1}[2]}(v) \cap NG_{G^{k+1}[2]}(u) \neq \emptyset\) and \([v, u(3)] \cap [d_3, e_3] = \emptyset\). By the induction hypothesis, there exist two disjoint paths \(S_1\) and \(S_2\) such that (1) \(S_1\) is a path joining \(v\) and \(u(3)\) with \(l(S_1) = l_1 - 1\); (2) \(S_2\) is a path joining \(d_3\) and \(e_3\) with \(l(S_2) = 4^k - 2 - l(S_1)\), and (3) \(S_1 \cup S_2\) spans \(G^{k+1}[3]\). Hence, \(P_1 = (u, u(3), S_1, v)\) and \(P_2 = (x, H, a_0, a(1), Q_1, b_1, b_2, R_1, d_2, d_3, e_2, e_3, R_2, c_2, c(1), Q_2, y)\) are the required paths as shown in Fig. 8.

For \(l_1 = 4^k - 1\), there is a Hamiltonian path \((u(3), S, v)\) in \(G^{k+1}[2]\) \{\(a(3)\)\} by Lemma 2. Hence, \(P_1 = (u, u(3), S, v)\) and \(P_2 = (x, H, a_0, a(3), a(1), Q_1, b_1, b_2, R, c_2, c(1), Q_2, y)\) are the required paths as shown in Fig. 9.

Case 1.3. \((4^k \leq l_1 \leq 2(4^k) - 1)\): For \(l_1 = 4^k\), there are two vertices \(a_2, b_0\) in \(NG_{G^{k+1}[2]}(u)\) and \(b_0\) in \(G^{k+1}[0]\) \{\(x\)\} such that \(a(1) \neq v, b(1) \neq y, b(3) \notin [v, a(3)]\). By Lemma 1, there are two Hamiltonian paths \((x, H, b_0)\) in \(G^{k+1}[0]\) and \((b_1, Q, y)\) in \(G^{k+1}[1]\). Since \(l(Q) = 4^k - 1\), \(Q\) can be written as \((b_1, Q_1, c_1, d_1, Q_2, y)\) for some vertices \(c_1\) and \(d_1\) such that \([c(1), d(1)] \cap [u(3), a(2)] = \emptyset\). By the induction hypothesis, there exist two disjoint paths \((u, R_1, a_2)\) and \((c_2, R_2, d_2)\) with \(l(R_1) = 1\) and \(l(R_2) = 4^k - 3\) such that \(R_1 \cup R_2\) spans \(G^{k+1}[2]\). By Lemma 2, there exists a Hamiltonian path \((a_3, S, v)\) in \(G^{k+1}[3]\) \{\(b_3\)\}. Hence, \(P_1 = (u, R_1, a_2, a_3, S, v)\) and \(P_2 = (x, H, b_0, b_3, b_1, Q_1, c_1, c_2, R_2, d_2, a(1), Q_2, y)\) are the required paths as shown in Fig. 10.
For $4^k + 1 \leq l_1 \leq 2(4^k) - 3$, by Lemma 4, $Q$ can be written as $\langle b(1), Q_1, c_1(1), d_1(1), Q_2, y \rangle$ for some vertices $c_1(1)$ and $d_1(1)$ such that $N_{G^{k+1}[2]}(u) \cap N_{G^{k+1}[2]}(a(2)) \neq \{c_2(2), d_2(2)\}$ and $\{u, a(2)\} \cap \{c_2(2), d_2(2)\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $\langle u, R_1, a(2) \rangle$ and $\langle c(2), R_2, d(2) \rangle$ with $l(R_1) = l_1 - 4^k$ and $l(R_2) = 4^k - 2 - l(R_1)$ such that $R_1 \cup R_2$ spans $G^{k+1}[2]$. By Lemma 1, there exists a Hamiltonian path $\langle a(3), S', v \rangle$ in $G^{k+1}[3]$. Hence, $P_1 = \langle u, R_1, a(2), a(3), S', v \rangle$ and $P_2 = \langle x, H, b(0), b(1), Q_1, c_1(1), c(2), R_2, d(2), d(1), Q_2, y \rangle$ are the required paths as shown in Fig. 11.
Fig. 11. Paths $P_1$ and $P_2$ constructed by Case 1.3 with $4^k + 1 \leq l_1 \leq 2(4^k) - 3$.

Fig. 12. Paths $P_1$ and $P_2$ constructed by Case 1.3 with $l_1 = 2(4^k) - 2$.

Fig. 13. Paths $P_1$ and $P_2$ constructed by Case 1.3 with $l_1 = 2(4^k) - 1$.

For $l_1 = 2(4^k) - 2$, there is a Hamiltonian path $\langle u, R, a_{(2)}, a_{(3)}, S, v \rangle$ in $G^{k+1}[2]$ by Lemma 1. Hence, $P_1 = \langle u, R, a_{(2)}, a_{(3)}, S, v \rangle$ and $P_2 = \langle x, H, b_{(0)}, b_{(3)}, c_{(1)}, Q, y \rangle$ are the required paths as shown in Fig. 12.

For $l_1 = 2(4^k) - 1$, $P_1 = \langle u, R, a_{(2)}, a_{(3)}, S', v \rangle$ and $P_2 = \langle x, H, b_{(0)}, b_{(1)}, Q, y \rangle$ are the required paths as shown in Fig. 13.
Hence, $P$ as shown in Fig. 14.

and $c$ such that $t$ is two disjoint paths $\langle \sigma^k_1(x), \sigma^k_1(y) \rangle = 1$: Without loss of generality, assume that $\sigma^k_1(x) = \sigma^k_1(y) = 1$ and $\sigma^k_1(v) = 2$ such that $u$ and $v$ are included in $G^{k+1}[1]$ and $G^{k+1}[2]$, respectively. Three subcases are necessary to be discussed depending on the values of $d_G(u, v)$ and $l_1$.

**Case 2.** $(\sigma^k_1(u) \neq \sigma^k_1(v)$ and $|[\sigma^k_1(u), \sigma^k_1(v)] \cap \{\sigma^k_1(x), \sigma^k_1(y)\}] = 1)$: Without loss of generality, assume that $\sigma^k_1(x) = \sigma^k_1(y) = 1$ and $\sigma^k_1(v) = 2$ such that $u$ and $v$ are included in $G^{k+1}[1]$ and $G^{k+1}[2]$, respectively. Three subcases are necessary to be discussed depending on the values of $d_G(u, v)$ and $l_1$.

**Case 2.1.** $(d_G(u, v) \leq l_1 \leq 4^k - 1$ with $d_G(u, v) = 1$): For $l_1 = 1$, we have $P_1 = \langle u, v \rangle$. There is a vertex $a_0$ in $G^{k+1}[0]\{x\}$ such that $a_0 \neq \{y, u\}$. By Lemma 1, there is a Hamiltonian path $x, H, a_0$ in $G^{k+1}[0]$. By Lemma 2, there is a Hamiltonian path $(a_1, Q, y)$ in $G^{k+1}[1]\{u\}$. Besides, $Q$ can be written as $(a_1, Q_1, b_1, c_1, Q_2, y)$ for some vertices $b_1$ and $c_1$. By Lemma 2, there exists a Hamiltonian path $(b_2, R, c_2)$ in $G^{k+1}[2]\{v\}$. Besides, $R$ can be written as $(b_2, R_1, d_2, e_2, R_2, c_2)$ for some vertices $d_2$ and $e_2$. By Lemma 1, there is a Hamiltonian path $(d_2, S, e_2)$ in $G^{k+1}[3]$. Hence, $P_2 = \langle x, H, a_0, a_1, Q_1, b_1, b_2, R_1, d_2, d_3, S, e_2, R_2, c_2, Q_2, y \rangle$, and $P_1$ and $P_2$ are the required paths as shown in Fig. 14.

For $l_1 = 2$, we have $P_1 = \langle u, u \rangle$. By Lemma 2, there is a Hamiltonian path $(d_2, S', e_2)$ in $G^{k+1}[3]\{u\}$. Hence, $P_2 = \langle x, H, a_0, a_1, Q_1, b_1, b_2, R_1, d_2, d_3, S', e_2, R_2, c_2, Q_2, y \rangle$, and $P_1$ and $P_2$ are the required paths as shown in Fig. 15.

For $3 \leq l_1 \leq 4^k - 1$, there are two vertices $a_0 \in N_{G^{k+1}[1]}(u)$ and $b_0 \in G^{k+1}[0].$ By the induction hypothesis, there exist two disjoint paths $(u, Q_1, a_1)$ and $(b_1, Q_2, y)$ with $l(Q_1) = 1$ and $l(Q_2) = 4^k - 3$ such that $Q_1 \cup Q_2$ spans $G^{k+1}[1].$ By Lemma 4, $Q_2$ can be written as $(b_1, c_1, d_1, Q_2, y)$ for some vertices $c_1$ and $d_1$ such that $N_{G^{k+1}[2]}(v) \cap N_{G^{k+1}[2]}(a_2) \neq \{c_2, d_2\}$ and $\{v, a_2\} \cap \{c_2, d_2\} = \emptyset.$ By the induction hypothesis, there exist two disjoint paths $(a_2, R_1, v)$ and $(c_2, R_2, d_2)$ with $l(R_1) = l_1 - 2$ and $l(R_2) = 4^k - 2 - l(R_1)$ such that $R_1 \cup R_2$ spans $G^{k+1}[2].$ $R_2$ can be written as $(c_2, R_2, e_2, f_2, R_2, d_2)$ for some vertices $e_2$ and $f_2.$ If $l(R_2) = 1$, then $e_2 = c_2$ and $f_2 = d_2$. By Lemma 1, there is a Hamiltonian path $(e_2, S, f_2)$ in $G^{k+1}[3].$ Hence, $P_1 = \langle u, Q_1, a_1, a_2, R_1, y \rangle$ and $P_2 = \langle x, H, b_0, b_1, Q_2, c_1, c_2, R_2, e_2, f_2, R_2, d_2, d_1, Q_2, y \rangle$ are the required paths as shown in Fig. 16.

Fig. 14. Paths $P_1$ and $P_2$ constructed by Case 2.1 with $l_1 = 1$.

Fig. 15. Paths $P_1$ and $P_2$ constructed by Case 2.1 with $l_1 = 2$.
By Lemma 4, the required paths as shown in Fig. 18.

\[ \langle s, a \rangle, \langle b, c \rangle, \langle d, e \rangle, \langle f, g \rangle \]

Case 2.2. \((d, \langle u, v \rangle) \leq l \leq 4^k - 1 \) with \(G_{k+1}[0] \leq v \leq 2^4 - 2\): For \(G_{k+1}[0] \leq v \leq 4^k - 2\), there is a vertex \(a(0) \in G_{k+1}[0] \langle x \rangle\) such that \(a(1) \neq \{y, u\}\). By Lemma 1, there are Hamiltonian paths \(x, H, a(0) \rangle \in G_{k+1}[0] \langle y \rangle\) and \(a(1), Q, y \rangle \in G_{k+1}[1] \langle u \rangle\). By Lemma 4, \(Q\) can be written as \(a(1), Q_1, b(1), c(1), Q_2, y \rangle\) for some vertices \(b(1)\) and \(c(1)\) such that \(N_{G_{k+1}[2]}(v) \cap N_{G_{k+1}[2]}(u) \neq \{b(2), c(2)\}\) and \(v, u \cap \{b(2), c(2)\} = \emptyset\). By the induction hypothesis, there exist two disjoint paths \(\langle u, R_1, v \rangle\) and \(\langle b(2), R_2, c(2) \rangle\) with \(l(R_1) = l_1 - 1\) and \(l(R_2) = 4^k - 2 - l(R_1)\) such that \(R_1 \cup R_2\) spans \(G_{k+1}[2]\). Besides, \(R_2\) can be written as \(\langle b(2), R_1, d(2), e(2), R_2, c(2) \rangle\) for some vertices \(d(2)\) and \(e(2)\). If \(l(R_2) = 1\), then \(d(2) = b(2)\) and \(e(2) = c(2)\). By Lemma 1, there is a Hamiltonian path \(d_3, S, e_3) \in G_{k+1}[3]\). Hence, \(P_1 = \langle u, u(2), R, v \rangle\) and \(P_2 = \langle x, H, a(0), a(1), Q_1, b(1), b(2), R_1, d(2), d(3), S, e_3, e(2), R_2, c(2), c(1), Q_2, y \rangle\) are the required paths as shown in Fig. 17.

For \(l_1 = 4^k - 1\), there is a Hamiltonian path \(\langle u(2), R, v \rangle\) in \(G_{k+1}[2] \langle c(2) \rangle\) by Lemma 2. By Lemma 1, there is a Hamiltonian path \(b(3), S', c(3) \rangle\) in \(G_{k+1}[3]\). Hence, \(P_1 = \langle u, u(2), R, v \rangle\) and \(P_2 = \langle x, H, a(0), a(1), Q_1, b(1), b(2), S', c(3), c(2), c(1), Q_2, y \rangle\) are the required paths as shown in Fig. 18.

Case 2.3. \((4^k \leq l \leq 2(4^k - 1))\): For \(l = 4^k\), there is \(a(1) \in N_{G_{k+1}[1]}(u) \setminus \{y\}\) such that \(a(2) \neq v\). Since \(\langle N_{G_{k+1}[1]}(u) \setminus N_{G_{k+1}[1]}(a(1)) \rangle = 2\), we have \(|N_{G_{k+1}[1]}(y) \setminus (N_{G_{k+1}[1]}(u) \setminus N_{G_{k+1}[1]}(a(1)))| \geq 3k - 2 \geq 4\). Thus, there is a vertex \(b(1) \in N_{G_{k+1}[1]}(y) \setminus (N_{G_{k+1}[1]}(u) \setminus N_{G_{k+1}[1]}(a(1)))\) such that \(b(0) \neq x\). By Lemma 1, there is a Hamiltonian path \(x, H, b(0) \rangle\) in \(G_{k+1}[0]\). By the induction hypothesis, there exist two disjoint paths \(\langle u, Q_1, a(1) \rangle\) and \(\langle b(1), Q_2, y \rangle\) with \(l(Q_1) = 1\) and \(l(Q_2) = 4^k - 3\) such that \(Q_1 \cup Q_2\) spans \(G_{k+1}[1]\). Since \(l(Q_2) = 4^k - 3\), \(Q_2\) can be written as \(\langle b(1), Q_2, c(1), d(1), Q_2, y \rangle\) for some vertices \(c(1)\) and \(d(1)\) such that \(d(2) \neq v\). By Lemma 2, there is a Hamiltonian path \(\langle a(2), R, v \rangle\) in \(G_{k+1}[2] \langle d(2) \rangle\). By Lemma 1, there is a Hamiltonian path \(\langle c(3), S, d(3) \rangle\) in \(G_{k+1}[3]\). Hence, \(P_1 = \langle u, Q_1, a(1), a(2), R, v \rangle\) and \(P_2 = \langle x, H, b(0), b(1), Q_2, c(1), c(3), S, d(3), d(2), Q_2, y \rangle\) are the required paths as shown in Fig. 19.

For \(4^k + 1 \leq l \leq 2(4^k - 3)\). Since \(N_{G_{k+1}[1]}(u) \setminus N_{G_{k+1}[1]}(a(1)) \neq \{b(1), y\}\), by the induction hypothesis, there exist two disjoint paths \(\langle u, Q_1, a(1) \rangle\) and \(\langle b(1), Q_2, y \rangle\) with \(l(Q_1) = l_1 - 4^k\) and \(l(Q_2) = 4^k - 2 - l(Q_1)\) such that \(Q_1 \cup Q_2\) spans
Fig. 18. Paths $P_1$ and $P_2$ constructed by Case 2.2 with $l_1 = 4^k - 1$.

Fig. 19. Paths $P_1$ and $P_2$ constructed by Case 2.3 with $l_1 = 4^k$.

Fig. 20. Paths $P_1$ and $P_2$ constructed by Case 2.3 with $4^k + 1 \leq l_1 \leq 2(4^k) - 3$.

$G^{k+1}[1]$. By Lemma 1, there is a Hamiltonian path $\langle a_{(2)}, R', v \rangle$ in $G^{k+1}[2]$. Hence, $P_1 = \langle u, Q, a_{(1)}, a_{(2)}, R', v \rangle$ and $P_2 = \langle x, H, b_{(0)}, b_{(1)}, Q_{22}, c_{(1)}, c_{(3)}, S, d_{(3)}, d_{(1)}, Q_{22}, y \rangle$ are the required paths as shown in Fig. 20.

For $l_1 = 2(4^k) - 2$, there are two vertices $a_{(1)}$ in $G^{k+1}[1]\{u, y\}$ and $b_{(0)}$ in $G^{k+1}[0]\{x\}$ such that $b_{(1)} \notin \{y, a_{(1)}\}$. By Lemma 1, there are Hamiltonian paths $\langle a_{(2)}, R', v \rangle$ in $G^{k+1}[2]$ and $\langle b_{(3)}, S', y_{(3)} \rangle$ in $G^{k+1}[3]$. By Lemma 2, there is a Hamiltonian path $\langle u, Q, a_{(1)} \rangle$ in $G^{k+1}[1]\{y\}$. Hence, $P_1 = \langle u, Q, a_{(1)}, a_{(2)}, R', v \rangle$ and $P_2 = \langle x, H, b_{(0)}, b_{(3)}, S', y_{(3)}, y \rangle$ are the required paths as shown in Fig. 21.
Three cases are necessary to be discussed depending on the values of \( n \) and \( H \) such that \( \sigma(G_k) \) can be written as \( (\sigma(G_k))_1 \) and \( (\sigma(G_k))_2 \).

**Case 3.** \((\sigma^{k+1}(u) \neq \sigma^{k+1}(v) \text{ and } |[\sigma^{k+1}(u), \sigma^{k+1}(v)] \cap [\sigma^{k+1}(x), \sigma^{k+1}(y)]| = 2)\): Without loss of generality, assume that \( \sigma^{k+1}(u) = \sigma^{k+1}(x) = 0 \) and \( \sigma^{k+1}(v) = \sigma^{k+1}(y) = 1 \) such that \( u \) and \( v \) are included in \( G^{k+1}[0] \) and \( G^{k+1}[1] \), respectively.

Three cases are necessary to be discussed depending on the values of \( d_C(u, v) \) and \( l_1 \).

**Case 3.1.** \((d_C(u, v) \leq l_1 \leq 4^k - 1 \text{ with } d_C(u, v) = 1)\): For \( l_1 = 1 \), we have \( P_1 = (u, v) \). There is a vertex \( a_0 \) in \( G^{k+1}[0] \) such that \( a_0 \neq y \). By Lemma 2, there are two Hamiltonian paths \((x, H, a_0)\) in \( G^{k+1}[0] \) and \((a_0, Q, y)\) in \( G^{k+1}[1] \). Consequently, there exist two disjoint paths \((a_0, R, c_1)\) and \((b_1, s, d_1)\) in \( G^{k+1}[2] \) which can be written as \((b_1, d_1)\) for some vertices \( d_1 \) and \( c_1 \). By Lemma 1, there is a Hamiltonian path \((b_1, d_1)\) in \( G^{k+1}[2] \) which can be written as \((b_2, d_2)\) for some vertices \( d_2 \) and \( c_2 \). By Lemma 2, there is a Hamiltonian path \((b_2, d_2, e_2)\) in \( G^{k+1}[2] \) which can be written as \((b_2, d_2, e_2)\) for some vertices \( d_2 \) and \( e_2 \). Hence, \( P_1 = (u, v) \) and \( P_2 = (x, H, a_0, a_1, Q, b_1, b_2, R_1, d_2, d_3, s, e_2, R_2, c_2, c_1, y) \) are the required paths as shown in Fig. 23.

For \( 3 \leq l_1 \leq 4^k - 1 \), there is a vertex \( f_0 \) in \( N_{G^{k+1}[0]}(y) \) such that \( f_0 \neq y \). Since \(|N_{G^{k+1}[0]}(x) - N_{G^{k+1}[0]}(y)| = 3k - 2 \geq 4 \), there is a vertex \( g_0 \) in \( N_{G^{k+1}[0]}(x) - N_{G^{k+1}[0]}(y) \) such that \( g_0 \neq y \). By Lemma 2, there exist two disjoint paths \((u, H, f_0)\) and \((x, H, g_0)\) with \( l(H_1) = l_1 - 2 \) and \( l(H_2) = 4^k - 2 - l(H_1) \) such that \( H_1 \cup H_2 \) spans \( G^{k+1}[0] \). Again, by the induction hypothesis, there exist two disjoint paths \((f_0, Q_1', y)\) and \((g_0, Q_2', y)\) with \( l(Q_1') = 1 \) and \( l(Q_2') = 4^k - 3 \) such that \( Q_1' \cup Q_2' \) spans \( G^{k+1}[1] \). Besides, \( Q_2' \) can be written...
Fig. 23. Paths $P_1$ and $P_2$ constructed by Case 3.1 with $l_1 = 1$.

Fig. 24. Paths $P_1$ and $P_2$ constructed by Case 3.1 with $l_1 = 2$.

Fig. 25. Paths $P_1$ and $P_2$ constructed by Case 3.1 with $3 \leq l_1 \leq 4^k - 1$.

ten as $\langle g(1), Q_2', b(1), c(1), Q_2', y \rangle$ for some vertices $b(1)$ and $c(1)$. Hence, $P_1 = \langle u, H_1, f(0), f_1, Q_1', v \rangle$ and $P_2 = \langle x, H_2, g(0), g(1), Q_1', b(1), b(2), R_1, d(2), d(3), S, e(3), c(2), R_2, c(2), c(1), Q_2', y \rangle$ are the required paths as shown in Fig. 25.

Case 3.2. ($d_G(u, v) \leq l_1 \leq 4^k - 1$ with $d_G(u, v) \geq 2$): For $l_1 = d_G(u, v)$ with $d_G(u, v) = 2$, we have $l_1 = 2$, which implies $N_G(u) \cap N_G(v) \neq \{x, y\}$. Since $N_G(u) \cap N_G(v) = \{u(1), v(0)\}$, we have $\{u(1), v(0)\} \neq \{x, y\}$. Without loss of generality, assume that $u(1) \neq y$. There are two vertices $a(1) \notin N_{G^{k+1}[1]}(u(1)) \cap N_{G^{k+1}[1]}(v)$ in $G^{k+1}[1]\{u(1), v, y, x(1)\}$, and $b(2)$ in $G^{k+1}[2]\{a(2)\}$. By the induction hypothesis, there exist two disjoint paths $\langle u(1), Q_1, v \rangle$ and $\langle a(1), Q_2, y \rangle$ with $l(Q_1) = l_1 - 1$ and $l(Q_2) =$
4^k - 2 - l(Q_1) such that Q_1 \cup Q_2 spans G^{k+1}[1]. By Lemma 2, there is a Hamiltonian path \langle x, H, a_0(1) \rangle in G^{k+1}[0]\{u\}. By Lemma 1, there are two Hamiltonian paths \langle a_2(1), R, b_2(1) \rangle in G^{k+1}[2] and \langle a_3(1), S, b_3(1) \rangle in G^{k+1}[3]. Hence, P_1 = \langle u, u(1), Q_1, v \rangle and P_2 = \langle x, H, a_0(1), S, b_3(1), b_2(1), R, a_2(1), a(1), Q_2, y \rangle are the required paths as shown in Fig. 26.

For l_1 = d_G(u, v) with d_G(u, v) \geq 3, if u(1) \neq y or v(0) \neq x, then the proof is the same as the situation that l_1 = d_G(u, v) with d_G(u, v) = 2. On the other hand, \langle u(1) = y \rangle and v(0) = x), there are four vertices a_0(1), b_0(1), c_1(1), and d_2(1) such that a_0(1) \in N_{G^{k+1}[0]}(u) and d_G(a_0(1), v) = d_G(u, v) = 1, b_0(1) \notin N_{G^{k+1}[0]}(u) \cap N_{G^{k+1}[1]}(a_0(1)) is included in G^{k+1}[0]\{u, a_0(1), x\}. By Lemma 2, there is a Hamiltonian path \langle a_0(1), Q_1, v_1, 2 \rangle spans G^{k+1}[1]. By Lemma 1, there are two Hamiltonian paths \langle c_2(1), R, d_2(1) \rangle in G^{k+1}[2] and \langle b_3(1), S, d_3(1) \rangle in G^{k+1}[3]. Hence, P_1 = \langle u, H_1, a_0(1), a_1(1), Q_1, v \rangle and P_2 = \langle x, H_2, b_0(1), b_3(1), S, d_3(1), d_2(1), R, c_2(1), c_1(1), Q_2, y \rangle are the required paths as shown in Fig. 27.

For d_G(u, v) + 1 \leq l_1 \leq 4^k - 1, there is a vertex a_0(1) in G^{k+1}[0]\{u, v\} such that a_0(1) \neq y. By Lemma 2, there are two Hamiltonian paths \langle x, H, a_0(1) \rangle in G^{k+1}[0]\{u\} and \langle a_1(1), Q, y \rangle in G^{k+1}[1]\{v\}. By Lemma 4, Q can be written as \langle a_0(1), Q_2, b_1(1), v_1(1), b_2(1), Q_2, y \rangle for some vertices b_1(1) and c_1(1) such that N_{G^{k+1}[2]}(u(2)) \cap N_{G^{k+1}[2]}(v_2(1)) \neq \emptyset. By the induction hypothesis, there exist two disjoint paths \langle u(2), R_1, v(2) \rangle and \langle b_1(2), R_2, c_2(2) \rangle with l(R_1) = l_1 - 2 and l(R_2) = 4^k - 2 - l(R_1) such that R_1 \cup R_2 spans G^{k+1}[2]. Besides, R_2 can be written as \langle b_2(2), R_2, d_2(2), c_2(2) \rangle for some vertices d_2(2) and e_2(2). By Lemma 1, there is a Hamiltonian path \langle d_3(2), S, e_3(2) \rangle in G^{k+1}[3]. Hence, P_1 = \langle u, u(2), R_1, v(2), y \rangle and P_2 = \langle x, H, a_0(1), a_1(1), Q_1, b(1), b_2(1), R_2, d_2(2), d_3(2), S, e_3(2), e_2(2), R_2, c_2(2), c_1(1), Q_2, y \rangle are the required paths as shown in Fig. 28.
Case 3.3. \((4^k \leq l_1 \leq 2(4^k) - 1)\): For \(l_1 = 4^k\), there are two vertices \(a(1) \in N_{G^{k+1}[1]}(v) - \{y\}\) and \(b(1) \in G^{k+1}[1]\{v, a(1), y\}\) such that \(a(0) \neq \{u, x\}\) and \(b(0) \neq x\). By Lemma 2., there is a Hamiltonian path \((u, H, a(0))\) in \(G^{k+1}[0]\{x\}\). By induction hypothesis, there exist two disjoint paths \((a(1), Q_1, v)\) and \((b(1), Q_2, y)\) with \(l(Q_1) = 1\) and \(l(Q_2) = 4^k - 3\) such that \(Q_1 \cup Q_2\) spans \(G^{k+1}[1]\) in which \(Q_2\) can be written as \((b(1), Q_{21}, c(1), d(1), Q_{22}, y)\). By Lemma 1, there are two Hamiltonian paths \((c(2), R, d(2))\) in \(G^{k+1}[2]\) and \((X_{23}, S, b(3))\) in \(G^{k+1}[3]\). Hence, \(P_1 = (u, H, a(0), a(1), Q_1, v)\) and \(P_2 = (x, x(3), S, b(3), b(1), Q_{21}, c(1), c(2), R, d(2), d(1), Q_{22}, y)\) are the required paths as shown in Fig. 29.

For \(4^k + 1 \leq l_1 \leq 2(4^k) - 3\), there exists a Hamiltonian path \((b(1), Q', y)\) in \(G^{k+1}[1]\{v\}\) by Lemma 2. By Lemma 4, \(Q'\) can be written as \((b(1), Q'_1, c(1), d(1), Q'_2, y)\) for some vertices \(c(1)\) and \(d(1)\) such that \(N_{G^{k+1}[2]}(a(2)) \cap N_{G^{k+1}[2]}(V(2)) \neq (2, d(2))\) and \([a(2), v(2)] \cap [c(2), d(2)] = \emptyset\). By the induction hypothesis, there exist two disjoint paths \((a(2), P_1, v(2))\) and \((c(2), R_2, d(2))\) with \(l(R_1) = l_1 - 4^k\) and \(l(R_2) = 4^k - 2 - l(R_1)\) such that \(R_1 \cup R_2\) spans \(G^{k+1}[2]\). Hence, \(P_1 = (u, H, a(0), a(2), R', v(2), y)\) and \(P_2 = (x, x(3), S, b(3), b(1), Q'_1, c(1), c(2), R_2, d(2), d(1), Q'_2, y)\) are the required paths as shown in Fig. 30.

For \(l_1 = 2(4^k) - 2\), there is a Hamiltonian path \((a(2), R', v(2))\) of \(G^{k+1}[2]\{v(2)\}\) by Lemma 2. Hence, \(P_1 = (u, H, a(0), a(2), R', v(2), v)\) and \(P_2 = (x, x(3), S, b(3), b(1), Q', y)\) are the required paths as shown in Fig. 31.

For \(l_1 = 2(4^k) - 1\), there is a Hamiltonian path \((a(2), R', v(2))\) in \(G^{k+1}[2]\) by Lemma 1. Hence, \(P_1 = (u, H, a(0), a(2), R', v(2), v)\) and \(P_2 = (x, x(3), S, b(3), b(1), Q', y)\) are the required paths as shown in Fig. 32.

Case 4. \(\{\sigma^{k+1}(u) = \sigma^{k+1}(v)\} \cap \{\sigma^{k+1}(x), \sigma^{k+1}(y)\} = \emptyset\): Without loss of generality, assume that \(\sigma^{k+1}(u) = \sigma^{k+1}(v) = 2\) such that \(u\) and \(v\) are both included in \(G^{k+1}[2]\).

For \(d_c(u, v) \leq l_1 \leq 4^k - 3\), there is a vertex \(a(0)\) in \(G^{k+1}[0]\{x\}\) such that \(a(1) \neq y\) and \(a(2) \neq \{u, v\}\). By Lemma 1, there is a Hamiltonian path \((x, H, a(0))\) in \(G^{k+1}[0]\) which can be written as \((x, H_1, b(0), c(0), H_2, a(0))\) for some vertices \(b(0)\) and \(c(0)\). By Lemma 1, there is a Hamiltonian path \((a(1), Q, y)\) in \(G^{k+1}[1]\). By Lemma 4, \(Q\) can be written as \((a(1), Q_1, d(1), e(1), Q_2, y)\) for some vertices \(d(1)\) and \(e(1)\) such that \(N_{G^{k+1}[2]}(u) \cap N_{G^{k+1}[2]}(v) \neq \{d_2, e_2\}\) and \(\{u, v\} \cap \{d_2, e_2\} = \emptyset\). By the induction hypothesis, there exist two disjoint paths \((u, R_1, v)\) and \((d_2, R_2, e_2)\) with \(l(R_1) = l_1\) and \(l(R_2) = 4^k - 2 - l(R_1)\) such
that $R_1 \cup R_2$ spans $G^{k+1}[2]$. By Lemma 1, there is a Hamiltonian path $(b(3), S, c(3))$ in $G^{k+1}[3]$. Hence, $P_1 = (u, R_1, v)$ and $P_2 = (x, H_1, b(0), b(3), S, c(3), c(0), H_2, a(0), a(1), Q_1, d(1), d(2), R_2, e(2), e(1), Q_2, y)$ are required paths as shown in Fig. 33.

For $l_1 = 4^k - 2$, there is a Hamiltonian path $(u, R, v)$ in $G^{k+1}[2]\{a(2)\}$ by Lemma 2. Hence, $P_1 = (u, R, v)$ and $P_2 = (x, H_1, b(0), b(3), S, c(3), c(0), H_2, a(0), a(2), a(1), Q, y)$ are the required paths as shown in Fig. 34.
For $l_1 = 4^k - 1$, there is a Hamiltonian path $\langle u, R', v \rangle$ in $G^{k+1}[2]$ by Lemma 1. Hence, $P_1 = \langle u, R', v \rangle$ and $P_2 = \langle x, H_1, b(0), b(3), S, c(3), c(0), H_2, a(0), a(1), Q, y \rangle$ are the required paths as shown in Fig. 35.

For $4^k \leq l_1 \leq 2(4^k) - 4$, obviously, path $R$ can be written as $\langle u, R_1, f(2), g(2), R_2, v \rangle$ for some vertices $f(2)$ and $g(2)$. By Lemma 4, $H$ can be written as $\langle x, H_1, b(0), c(0), H_2, a(0) \rangle$ for some vertices $b(0)$ and $c(0)$ such that $N_{G^{k+1}[3]}(f(3)) \cap N_{G^{k+1}[3]}(g(3)) \neq \{b(3), c(3)\}$ and $\{f(3), g(3)\} \cap \{b(3), c(3)\} = \emptyset$. By the induction hypothesis, there exist two disjoint paths $\langle f(3), S_1, g(3) \rangle$ and $\langle b(3), S_2, c(3) \rangle$ with $l(S_1) = l_1 - 4^k + 1$ and $l(S_2) = 4^k - 2 - l(S_1)$ such that $S_1 \cup S_2$ spans $G^{k+1}[3]$. 
Hence, $P_1 = \langle u, R_1, f(2), f(3), S_1, g(3), g(2), R_2, v \rangle$ and $P_2 = \langle x, H_1, b(0), b(3), S_2, c(3), c(0), H_2, a(0), a(2), a(1), Q, y \rangle$ are the required paths as shown in Fig. 36.

For $l_1 = 2(4^k) - 3$, there is a Hamiltonian path $(f(3), S', g(3))$ in $G_{k+1}[3] \setminus \{a(3)\}$ by Lemma 2. Hence, $P_1 = \langle u, R_1, f(2), f(3), S', g(3), g(2), R_2, v \rangle$ and $P_2 = \langle x, H, a(0), a(3), a(2), a(1), Q, y \rangle$ are the required paths as shown in Fig. 37.

For $l_1 = 2(4^k) - 2$, there exist a Hamiltonian path $(f(3), S'', g(3))$ in $G_{k+1}[3]$ by Lemma 1. Hence, $P_1 = \langle u, R_1, f(2), f(3), S'', g(3), g(2), R_2, v \rangle$ and $P_2 = \langle x, H, a(0), a(2), a(1), Q, y \rangle$ are the required paths as shown in Fig. 38.
For $l_1 = 2(4^k) - 1$, obviously, path $R'$ can be written as $(u, R'_1, f_2, g_2, R'_2, v)$ for some vertices $f_2$ and $g_2$. Hence, $P_1 = (u, R'_1, f_2, g_2, R'_2, v)$ and $P_2 = (x, H, a_0, (a_1, Q, y)$ are the required paths as shown in Fig. 39.

**Case 5.** ($\sigma^{k+1}(u) = \sigma^{k+1}(v)$ and $|\sigma^{k+1}(u) \cap \sigma^{k+1}(v)| = 1$): Without loss of generality, assume that $\sigma^{k+1}(u) = \sigma^{k+1}(v) = 0$ such that $u$ and $v$ are both included in $G^{k+1}[0]$. 

For $d_G(u, v) \leq l_1 \leq 4^k - 3$, since $|N_{G^{k+1}[0]}(u) - (N_{G^{k+1}[0]}(u) \cap N_{G^{k+1}[0]}(v))| \geq 4$, there is a vertex $a_0 \in N_{G^{k+1}[0]}(x) - (N_{G^{k+1}[0]}(u) \cap N_{G^{k+1}[0]}(v))$ in $G^{k+1}[0][u, v]$ such that $d_1(1) = y$ and $N_{G^{k+1}[0]}(u) \cap N_{G^{k+1}[0]}(v) \neq \{x, a_0\}$. By the induction hypothesis, there exist two disjoint paths $(u, H_1, v)$ and $(x, H_2, a_0)$ with $l(H_1) = l_1$ and $l(H_2) = 4^k - 2 - l(H_1)$ such that $H_1 \cup H_2$ spans $G^{k+1}[0]$. By Lemma 1, there is a Hamiltonian path $(a_1, Q, y)$ in $G^{k+1}[1]$ which can be written as $(a_1, Q_1, b_1, c_1, Q_2, y)$ for some vertices $b_1$ and $c_1$. By Lemma 1, there is a Hamiltonian path $P_1$ in $G^{k+1}[2]$ joining $b_2$ and $c_2$, and $R$ can be written as $(b_2, R_1, d_2, e_2, R_2, c_2)$ for some vertices $d_2$ and $e_2$. Again, by Lemma 1, there is a Hamiltonian path $(d_3, S, e_3)$ in $G^{k+1}[3]$. Hence, $P_1 = (u, H_1, v)$ and $P_2 = (x, H_2, a_0, a_1, Q_1, b_1, c_1, Q_2, y)$ are the required paths as shown in Fig. 40.

For $l_1 = 4^k - 2$, there is a Hamiltonian path $(u, H, v)$ in $G^{k+1}[0][x]$ by Lemma 2. By Lemma 1, there is an Hamiltonian path $(x_3, S', a_3)$ in $G^{k+1}[3]$. Hence, $P_1 = (u, H, v)$ and $P_2 = (x, x_3, S', a_3, Q_1, b_1, c_1, Q_2, y)$ are the required paths as shown in Fig. 41.

For $4^k - 1 \leq l_1 \leq 2(4^k) - 5$, $l(H_1)$ and $l(H_2)$ can be reset as $4^k - 3$ and 1, and $H_1$ can be written as $(u, H_{11}, f_0, g_0, H_{12}, v)$ for some vertices $f_0$ and $g_0$. By Lemma 4, $R$ can be written as $(b_2, R_1, d_2, e_2, R_2, c_2)$ for some vertices $d_2$ and $e_2$ such that $N_{G^{k+1}[3]}(f_2) \cap N_{G^{k+1}[3]}(g_3) \neq \{d_2, e_3\}$ and $l(S_2) = 4^k - 2 - l(S_1)$ such that $S_1 \cup S_2$ spans $G^{k+1}[3]$. Hence, $P_1 = (u, H_{11}, f_0, S_1, g_0, H_{12}, v)$ and $P_2 = (x, H_2, a_0, a_1, Q_1, b_1, c_1, Q_2, y)$ are the required paths as shown in Fig. 42.
For $l_1 = 2(4^k) - 4$, there is a Hamiltonian path $(f(3), S''', g(3))$ in $G^{k+1}[3]\{a(3)\}$ by Lemma 2. Hence, $P_1 = (u, H_{11}, f_0), f(3), S'', g(3), g(0), H_{12}, v)$ and $P_2 = (x, H_2, a_0, a_3, a_1, Q_1, b_1, b_2, R, c_2, c_1, Q_2, y)$ are the required paths as shown in Fig. 43.

For $l_1 = 2(4^k) - 3$, there is a Hamiltonian path $(f(3), S''', g(3))$ in $G^{k+1}[3]$ by Lemma 1. Hence, $P_1 = (u, H_{11}, f_0, f(3), S'', g(3), g(0), H_{12}, v)$ and $P_2 = (x, H_2, a_0, a_1, Q_1, b_1, b_2, R, c_2, c_1, Q_2, y)$ are the required paths as shown in Fig. 44.
For $l_1 = 2(4^k) - 2$, since $l(H) = 4^k - 2$, path $H$ can be written as $\langle u, H_1, f_0, g_0, H_2, v \rangle$ for some vertices $f_0$ and $g_0$ such that $g_0(2) \neq a_2$. By Lemma 1, there is a Hamiltonian path $\langle x(2), R', a_2(2) \rangle$ in $G^{k+1}[2]$. Hence, $P_1 = \langle u, H_1, f_0, f_3, S''', g_3, g_0, H_2, v \rangle$ and $P_2 = \langle x, x(2), R', a_2, a_1, \rangle, Q, y \rangle$ are the required paths as shown in Fig. 45.

For $l_1 = 2(4^k) - 1$, there is a Hamiltonian path $\langle x(2), R'', a_2(2) \rangle$ in $G^{k+1}[2] \setminus \{g_0(2)\}$ by Lemma 2. Hence, $P_1 = \langle u, H_1, f_0, f_3, S''', g_3, g_0, H_2, v \rangle$ and $P_2 = \langle x, x(2), R'', a_2, a_1, \rangle, Q, y \rangle$ are the required paths as shown in Fig. 46. □
Lemma 6. If $G(4, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, then $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, where $r \geq 2$ and $m_i \geq 4$ for all $1 \leq i \leq r$.

Proof. This lemma is proved by induction on $m_r$. By the assumption, the basis step, i.e., $m_r = 4$, holds directly. Suppose that $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property for all $4 \leq m_r \leq k$. In the following, we prove that $G(k+1, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property. Let $G$ denote $G(k+1, m_{r-1}, \ldots, m_1)$ for brevity. Without loss of generality, assume that $l_1 \leq l_2$, and thus $l_1 \leq ((k+1) \times N_r - 2)/2$. Moreover, $l_1 \leq ((k+1) \times N_r - 2) - \max(k, d_G(x, y))$, as explained below. If $k \geq d_G(x, y)$, then $l_1 \leq ((k+1) \times N_r - 2) - k$ because $((k+1) \times N_r - 2)/2 \leq (k \times N_r - 2) - k$ can be assured by $k \geq 4$, $N_r \geq 4' - 1$, and $r \geq 2$. On the other hand ($k < d_G(x, y)$), we have $l_1 \leq ((k \times N_r - 2) - d_G(x, y)$ because $((k+1) \times N_r - 2)/2 \leq (k \times N_r - 2) - d_G(x, y)$ can be assured by $k \geq 4$, $N_r \geq 4' - 1$, $d_G(x, y) \leq r$, and $r \geq 2$.

Let $u, v, x$ and $y$ be four distinct vertices in $G$. Since $G'[0], G'[1], \ldots, G'[k]$ are vertex-disjoint to each other, there exists a $G'[j]$ such that none of $u, v, x$ and $y$ is included in $G'[j]$, where $0 \leq j \leq k$. Without loss of generality, assume that $j = k$.

Since $d_G(u, v) \leq l_1, d_G(u, v) \leq l_1 \leq ((k+1) \times N_r - 2) - \max(k, d_G(x, y))$. The following discussions first exclude the situation that $l_1 = 2$ with $N_{G-G'[k]}(u) \cap N_{G-G'[k]}(v) = \{x, y\}$ and $N_{G}(u) \cap N_{G}(v) = \{x, y\}$. Since $G - G'[k] \cong G(k, m_{r-1}, \ldots, m_1)$, by the induction hypothesis, there exist two disjoint paths $(u, H_1, v)$ and $(x, H_2, y)$ with $l(H_1) = l_1$ and $l(H_2) = (k \times N_r - 2) - l_1$ such that $H_1 \cup H_2$ spans $G - G'[k]$. Since $l(H_2) \geq k$, by Lemma 3, there exists an $m$-edge in $H_2$ where $m \neq r$. Thus, $H_2$ can be written as $(x, H_2, a, b, H_2, y)$ for some vertices $a$ and $b$ such that $a$ and $b$ is connected by an $m$-edge and $a(b) \neq b(a)$. By Lemma 1, there is a Hamiltonian path $(a(b), b(a))$ in $G'[k]$. Hence $P_1 = (u, H_1, v)$ and $P_2 = (x, H_2, a, b, b(a), b, H_2, y)$ are the required paths as shown in Fig. 47.

The rest of this proof considers the situation that $l_1 = 2$ with $N_{G-G'[k]}(u) \cap N_{G-G'[k]}(v) = \{x, y\}$ and $N_G(u) \cap N_G(v) = \{x, y\}$. Since $N_{G-G'[k]}(u) \cap N_{G-G'[k]}(v) \subseteq N_G(u) \cap N_G(v)$ and $u$ and $v$ are not in $G'[k]$, there is a vertex $t(k) \in N_G(u) \cap N_G(v)$ in $G'[k]$ such that $u, v, t \in G(t(k)) = \{x, y\}$ and $u$ and $v$ are $r$-neighbors. Therefore, $u(t(k)) = v(t(k)) = t(k)$ and $d_G(u, v) = 2$. Since $N_{G-G'[k]}(u) \cap N_{G-G'[k]}(v) = \{x, y\}$ and $u$ and $v$ are $r$-neighbors, only four vertices $u, v, x$ and $y$ are connected by $r$-edges in $G - G'[k]$. Then $\sigma(x) \neq \sigma'(x), y = \sigma'(y) = \sigma(y) = 2$, and $\sigma'(v) = 3$, such that $x, y, u$ and $v$ are included in $G'[0], G'[1], G'[2], \ldots, G'[3]$, respectively. There is a vertex $a(0) \in G'[0]\{x\}$ such that $a(0) \neq y$. By Lemma 1, there are two Hamiltonian paths $(x, H, a(0))$ in $G'[0]$ and $(a(1), y, G'[1])$. Since $l(G) \geq 4r - 1 - 1$, $G$ can be written as $(a(1), Q, b(1), c(1), Q, y)$ for some vertices $b(1)$ and $c(1)$ such that $b(1), c(1) \cap \{u\} = \emptyset$. By Lemma 2, there is a Hamiltonian path $(b(2), R, c(2))$ in $G'[2]\{u\}$ such that $b(2), R, c(2)$ are vertices $b(2), c(2), e(2)$, $R, c(2)$ for some vertices $b(2)$ and $c(2)$. By Lemma 2, there is a Hamiltonian path $(d(3), e(3))$ in $G'[3]\{v\}$ which can be written as $(d(3), S, e(3))$, $d(3), e(3)$ for some vertices $f(3)$ and $g(3)$. Again, by Lemma 2, there is a Hamiltonian path $(f(t(4)), T, g(4))$ in $G'[4]\{u(4)\}$. Hence, $P_1 = (u, t(4), v)$ and $P_2 = (x, H, a(0), a(1), Q, b(1), b(2), R, d(2), d(3), S, f(3), f(t(4)), T, g(4), g(3), e(3), e(2), R, e(2), c(2), c(1), Q, y)$ are the required paths as shown in Fig. 48.

Theorem 1. $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, where $m_i \geq 4$ for all $1 \leq i \leq r$.

Proof. By Lemmas 5 and 6, $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property for $m_i \geq 4$ and $r \geq 2$. Since $G(m_r, m_{r-1}, \ldots, m_1)$ is isomorphic to $G(m_r-1, m_{r-2}, \ldots, m_1, m_r)$, $G(m_r, m_{r-1}, \ldots, m_1)$ is isomorphic to $G(4, 4, \ldots, 4, m_1)$ when $m_r = m_1$. Thus, $G(4, 4, \ldots, 4, m_1)$ satisfies the 2RP-property for $m_1 \geq 4$. By Lemma 6 again, $G(m_1, m_4, \ldots, m_1)$ also satisfies the 2RP-property for $m_1 \geq 4$. This further implies that $G(4, 4, 4, 4, m_1, m_1)$ satisfies the 2RP-property for $m_1 \geq 4$. By applying Lemma 6 repeatedly, we can find that all of $G(4, 4, 4, 4, m_1, m_1), G(4, 4, 4, 4, m_1, m_1, m_1, m_1, \ldots)$, and $G(m_r, m_{r-1}, \ldots, m_1)$ satisfy the 2RP-property, where $m_i \geq 4$ for all $1 \leq i \leq r$. This completes the proof.

4. Conclusion and future work

The 2RP-property of an interconnection network indicates the path embedding capability of the network. This work first proves that $G(m_i, m_{i-1}, \ldots, m_1)$ is 1-Hamiltonian-connected, where $m_i \geq 3$ for all $1 \leq i \leq r$. Then, by using this
1-Hamiltonian-connected property, our study shows that $G(m_r, m_{r-1}, \ldots, m_1)$ satisfies the 2RP-property, where $m_i \geq 4$ for all $1 \leq i \leq r$. As a further study, it is interesting to investigate the 3RP-property of a graph $G$. That is, let $\langle u, v \rangle$, $\langle w, x \rangle$ and $\langle y, z \rangle$ be three pairs of distinct vertices and $l_1, l_2$ and $l_3$ be three integers with $l_1 + l_2 + l_3 = |V(G)| - 3$. A graph satisfies the 3RP-property if there exist three disjoint paths $P_1$, $P_2$ and $P_3$ such that (1) $P_1$ is a path joining $u$ to $v$ with $l(P_1) = l_1$; (2) $P_2$ is a path joining $w$ to $x$ with $l(P_2) = l_2$; (3) $P_3$ is a path joining $y$ to $z$ with $l(P_3) = l_3$, and (4) $P_1 \cup P_2 \cup P_3$ spans $G$.

Acknowledgements

The authors would like to thank anonymous referees for their careful reading with corrections and useful comments which helped to improve the paper. Besides, the authors would like to thank the National Science Council of the Republic of China for financially supporting this research under Contract Nos. NSC 99-2221-E-260-010- and NSC 101-2221-E-146-014-.

References

