Hamiltonian cycles in hypercubes with faulty edges

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Article info

Article history:
Received 17 November 2010
Received in revised form 1 August 2013
Accepted 3 September 2013
Available online 11 September 2013

Keywords:
Hypercube
Hamiltonian cycle
Hamiltonian laceable
Fault tolerant

1. Introduction

A path (respectively, cycle) in a graph G is a Hamiltonian path (respectively, Hamiltonian cycle) if every vertex in G appears exactly once in the path (respectively, cycle). A graph is Hamiltonian if there exists a Hamiltonian cycle in it. The Hamiltonian problem which is to determine whether a graph is Hamiltonian is one of the most popular problems. In interconnection networks, hypercubes are widely studied [6–8,21] and play an important role in parallel computations [15,16]. A number of other topologies, such as paths, trees, rings, and meshes, can be embedded into a hypercube. There are a lot of studies on the fault-tolerant in hypercubes [1,2,4,5,9–11,14,19]. In this paper, we are concerned with the Hamiltonian problem in hypercubes with faulty edges.

In general, there exist more than one Hamiltonian cycle in a hypercube. In [3], Chan and Lee studied the problem of finding a Hamiltonian cycle in an n-dimensional hypercube, denoted by Qn, with a set F of faulty edges. They showed that there exists a Hamiltonian cycle in any Qn − F with |F| ≤ n − 2, where Qn − F denotes the resulting graph after removing all edges in F from Qn. They also proved that there exists a fault-free Hamiltonian cycle in Qn for n ≥ 3 and |F| ≤ 2n − 5 under the constraint that the degree of every vertex in Qn − F is at least 2. This result is optimal since there exists a hypercube with 2n − 4 faulty edges in which no fault-free Hamiltonian cycle exists. In this paper, we show that the maximum number of faulty edges can be extended to 3n − 8 for n ≥ 5 such that there still exists a Hamiltonian cycle in Qn − F under the following two constraints:

(1) the degree of every vertex is at least two (see Fig. 1(a)), and
(2) there do not exist a pair of nonadjacent vertices in a 4-cycle whose degrees are both two (see Fig. 1(b)).
It is easy to verify that if $Q_n$ contains a 4-cycle as shown in Fig. 1(b), then there is no fault-free Hamiltonian cycle in $Q_n$. For example, if vertices $a$, $u$, and $b$ form a subpath of a cycle, then vertex $v$ cannot be a vertex in the cycle when the length of the cycle is greater than 4. The graph in Fig. 1(b) is also mentioned in [3] to prove that their result is optimal.

The rest of this paper is organized as follows. In Section 2, we describe some important properties in hypercubes. We prove our main result in Section 3. Concluding remarks and future work are given in Section 4.

2. Preliminaries

An $n$-dimensional hypercube $Q_n$, also called $n$-cube, can be modeled as a graph with vertex set $V(Q_n)$ and edge set $E(Q_n)$. In $Q_n$, there are $2^n$ vertices and $2^{n-1}$ edges. Each vertex $u$ of $Q_n$ can be labeled by an $n$-bit string $b_{n-1}b_{n-2}...b_1b_0$. We use $u^{(i)}$ to denote the binary string $b_{n-1}b_{n-2}...b_{i+1}b_{i}...b_0$ for $0 \leq i \leq n-1$ and we call the edge $uu^{(i)}$ an $i$-dimensional edge. Thus if vertices $u$ and $v$ are adjacent, then $u = v^{(i)}$ and $v = u^{(i)}$ for some $0 \leq i \leq n-1$. If $u = uu^{(i)}$, then $v^{(i)} = (u^{(i)})^{(i)}$ is simplified as $u^{(i)}$. We also use $d_i(u)$ to stand for the edge $uu^{(i)}$. Let $D_i = \{d_i(u) | u \in V(Q_n)\}$, i.e., the set containing all $i$-dimensional edges of $Q_n$. It is clear that $|D_i| = 2^{n-1}$ for every $0 \leq i \leq n-1$.

We use $Q^L_{n-1}$ (respectively, $Q^R_{n-1}$) to denote the subgraph of $Q_n$ induced by the vertices in $\{u \in V(Q_n) | b_i = 0\}$ (respectively, $\{u \in V(Q_n) | b_i = 1\}$) for $0 \leq i \leq n-1$. Thus $Q_n$ can be decomposed to $Q^L_{n-1}$ and $Q^R_{n-1}$ at any dimension $i$ (see Fig. 2 for an illustration). Furthermore, both $Q^L_{n-1}$ and $Q^R_{n-1}$ are $(n-1)$-cubes. $Q^L_{n-1}$ and $Q^R_{n-1}$ are simply written as $Q^{L_{n-1}}$ and $Q^{R_{n-1}}$, respectively, when dimension $i$ is clear from the context. In order to classify the faulty edges in $Q_n$, we use $f(Q_n)$ and $f(D_i)$ to denote the number of faulty edges in $Q_n$ and the number of faulty edges in the set $D_i$, respectively. In the rest of this paper, we assume that $Q_n$ is decomposed to $Q^L_{n-1}$ and $Q^R_{n-1}$ at dimension 0 and $f(Q^L_{n-1}) \geq f(Q^R_{n-1})$ unless otherwise stated. If there exists a vertex $u$ in $Q_n$ which is incident with only two nonfaulty edges, say $d_i(u)$ and $d_{i+1}(u)$, and $Q_n$ is decomposed to $Q^L_{n-1}$ and $Q^R_{n-1}$ at dimension $h$ (or $k$), then we call $u$ a pivot vertex in the $(n-1)$-cube to which $u$ belongs (see Fig. 1(a)).

A path $P$ of length $k$ from vertex $x$ to vertex $y$ in $Q_n$ is a sequence of distinct vertices $v_0, v_1, ..., v_k$ in which $v_0 = x$, $v_k = y$, and $v_i, v_{i+1} \in E(Q_n)$, for $i = 0, 1, ..., k-1$ and $k \geq 1$. We also use $\langle v_0, P, v_k \rangle$ to denote $P$ in order to indicate the two endpoints $v_0$ and $v_k$ of $P$. For consistency, an edge $uv$ can also be represented as a path $(u, v)$. For two vertex-disjoint paths $\langle x, P, y \rangle$ and $\langle u, Q, v \rangle$ in which $y$ and $u$ are adjacent, we use $\langle x, P, y, u, Q, v \rangle$ to denote the concatenation of paths $P$ and $Q$. A cycle is also a sequence of distinct vertices $v_0, v_1, ..., v_k$ with $k \geq 3$ except $v_0 = v_k$. In $Q_n - F$, a cycle of length 4 is called a forbidden cycle (f-cycle for short) if it has two nonadjacent vertices of degree 2. It is obvious that if there exists an f-cycle in $Q_n - F$, then there is no Hamiltonian cycle in $Q_n - F$. In [3], Chan and Lee proved that if Constraint (1) is satisfied, then there exists a fault-free Hamiltonian cycle in $Q_n, n \geq 3$, when there are at most $2n - 5$ faulty edges. If Constraint (1) is not satisfied, then the number of faulty edges can only be at most $n - 2$ as shown in the following lemma.

![Fig. 1. Illustrations for Constraints (1) and (2) in which faulty edges are represented by dashed lines.](image1)

![Fig. 2. $Q_n$ is decomposed to $Q^L_{n-1}$ and $Q^R_{n-1}$ at dimension 3.](image2)
Lemma 1 [3]. For any n-cube (n $\geq 3$) with at most n – 2 faulty edges, there still exists a fault-free Hamiltonian cycle.

A graph $G$ is bipartite if its vertex set can be partitioned into two disjoint partite sets $V_0(G)$ and $V_1(G)$ such that there is no edge between the vertices in $V_0(G)$ and there is no edge between the vertices in $V_1(G)$. We say that two vertices $u$ and $v$ are of the same color if and only if $u$ and $v$ are in the same partite set. A bipartite graph $G = (V_0 \cup V_1, E)$ is said to be equitable if and only if $|V_0| = |V_1|$. Simmons introduced the concept of Hamiltonian laceability for equitable bipartite graphs [17]. An equitable bipartite graph $G = (V_0 \cup V_1, E)$ is Hamiltonian laceable if there exists a Hamiltonian path between any two vertices $x \in V_0$ and $y \in V_1$ [13]. Harary and Hayes proved that $Q_n$ for $n \geq 3$ is Hamiltonian laceable [12]. In [20], Tsai, Tan, Liang, and Hsu proved that for any n-cube $(n \geq 3)$ with at most $n – 2$ faulty edges is Hamiltonian laceable. Tsai showed that the number of maximal number of tolerable faulty edges can be extended to $2n – 5$ if Constraint (1) is satisfied [18]. A result in [18] is described as follows.

Theorem 2 [18]. For any $Q_n$ with $n \geq 3$ and $|F| \leq 2n – 5$, if Constraint (1) is satisfied, then $Q_n – F$ is Hamiltonian laceable.

Theorem 3 [18]. Suppose that $u$, $v$, $x$, and $y$ are four distinct vertices of $Q_n$ for $n \geq 2$ and $u$ and $v$ (respectively, $x$ and $y$) are with different colors. Then there exist two vertex-disjoint paths $\langle u, P_1, v \rangle$ and $\langle x, P_2, y \rangle$ which span $V(Q_n)$.

3. Main results

We shall prove Theorem 7 by induction on $n$. The basis step of Theorem 7, i.e., $n = 5$, is proved in Lemma 6. The reason why the basis is not from $n = 4$ is that there exists a $Q_4$ as shown in Fig. 3 in which no Hamiltonian cycle exists in $Q_4 – F$. However, for proving Lemma 6, we still need to figure out what the conditions are when there exists a Hamiltonian cycle in $Q_4 – F$ with $|F| = 4$. These conditions are described in Lemmas 4 and 5. In the following, we say that an edge is a replacement edge (r-edge for short) if it is not adjacent to any faulty edge in dimension 0. Note that an r-edge itself might also be a faulty edge. For a vertex incident with $n – 2$ faulty edges, the two nonfaulty edges incident to it are called good edges (g-edges for short).

Lemma 4. Let $Q^L_{4,3}$ and $Q^R_{4,3}$ be two subcubes of $Q_4$ with $|F| \leq 4$ and $f(Q^R_{4,3}) \leq 1$ for some $0 \leq i \leq 3$. Let $C$ be a Hamiltonian cycle in $Q^L_{4,3}$. If there is at most one faulty edge in $C$ and the faulty edge, if exists, is an r-edge, then there exists a fault-free Hamiltonian cycle in $Q_4$.

Proof. It suffices to consider the case where there is a faulty r-edge, say $e = \alpha \beta$, in $C$. Since $f(Q^R_{4,3}) \leq 1$, there is a fault-free Hamiltonian path $\langle \alpha^0, \mathcal{R}, \beta^0 \rangle$ in $Q^R_{4,3}$ by Theorem 2. Thus a fault-free Hamiltonian cycle $\langle x, C – e, \beta, \beta^0, \mathcal{R}, \alpha^0, x \rangle$ can be constructed in $Q_4$ where $C – e$ denotes removing the edge $e$ from $C$ (see Fig. 4). \qed

Lemma 5. Let $F$ be the faulty edge set of $Q_4$ with $|F| \leq 4$ and $f(D_i) < 4$ for $0 \leq i \leq 3$. If Constraints (1) and (2) are satisfied, then there is a Hamiltonian cycle in $Q_4 – F$.

Proof. We only need to consider the case where $|F| = 4$. Let $F = \{f_0, f_1, f_2, f_3\}$ be the set of faulty edges in $Q_4$. We may assume without loss of generality that $f_0$ lies in dimension 0. Therefore, we decompose $Q_4$ into two subcubes, $Q^L_4$ and $Q^R_4$, at dimension 0. Note that $f(Q^L_{4,3}) \geq f(Q^R_{4,3})$. By taking into consideration the possible values of $f(Q^L_{4,3}), f(D_0)$, and $f(Q^R_{4,3})$, there are three cases to consider.

Case 1: $f(Q^L_{4,3}) = 3, f(D_0) = 1$, and $f(Q^R_{4,3}) = 0$. In this case, $f_0$ lies in dimension 0, and $f_1, f_2, f_3$ are in $Q^R_4$. By Lemma 1, there exists a Hamiltonian cycle $C$ in $Q^L_{4,3} – \{f_1\}$. Note that $C$ might contain $f_2$ and $f_3$ but $f_1$. There are three subcases to consider.

Subcase 1.1: Both $f_2$ and $f_3$ are not in $C$.

Since $C$ has no faulty edge and $f(D_0) = 1$, there are at least six r-edges in $C$. By Lemma 4, a Hamiltonian cycle can be constructed in $Q_4 – F$.

Subcase 1.2: Exactly one of $f_2$ and $f_3$ is in $C$.

Fig. 3. A $Q_4$ with four faulty edges has no Hamiltonian cycle.
Assume that $f_2 = x\beta$ is in $C$. If $f_2$ is a faulty $r$-edge, then, by Lemma 4, a Hamiltonian cycle can be constructed in $Q_4 - R$. It remains to consider the case where $f_2$ is adjacent to $f_0$. Assume that $f_0 = d_0(x)$ and $f_2 = d_1(x)$ (see Fig. 5(a)), both $d_2(x)$ and $d_3(x)$ are $g$-edges since each of them is incident with two nonfaulty edges. Assume that $d_3(x) \not\in C$. Then, we can divide $C$ into four subpaths: $(\alpha, \beta)$, $(\beta, P_1, x(3))$, $(x(3), \rho)$, and $(\rho, P_2, x)$. Since $f(Q^4_n) = 0$, by Theorem 2, the cycle $(\beta, P_1, x, P_2, \rho, \rho(0), R, \beta(0), \beta)$ forms a Hamiltonian cycle in $Q_4 - R$.

**Subcase 1.3:** Both $f_2$ and $f_3$ are in $wC$.

Let $f_2 = x\beta$ and $f_3 = \gamma\delta$. Accordingly, $C$ can be divided into four subpaths: $(\alpha, \beta)$, $(\beta, P_1, \gamma)$, $(\gamma, \delta)$, and $(\delta, P_1, x)$ (see Fig. 5(b)). If both $f_2$ and $f_3$ are $r$-edges, by Theorem 3, a Hamiltonian cycle $(\gamma, P_2, \beta, \beta(0), R_1, x(0), x, P_1, \delta, \delta(0), R_2, \gamma(0), \gamma)$ can be constructed in $Q_4 - R$.

If one of $f_2$ and $f_3$, say $f_2$, is adjacent to $f_0$, then we may assume without loss of generality that $f_0 = d_0(x), f_2 = d_1(x)$, and $e = d_3(x) \not\in C$ is a $g$-edge. If $f_3 = x(3)\gamma$ (i.e., $x(3) = \gamma$) then, by using a similar construction as in Subcase 2.2, the cycle $(\beta, P_1, x, \gamma, P_2, \delta, \delta(0), R_1, \beta(0), \beta)$ is a Hamiltonian cycle in $Q_4 - R$ (see Fig. 5(a) by replacing $\rho$ and $\rho(0)$ by $\delta$ and $\delta(0)$, respectively).

If $f_3 \not= x(3)\gamma$, then $f_2$ can be in either $P_1$ or $P_2$. We only consider the case that $f_2$ is in $P_1$. The other case can be handled similarly. By Theorem 3, a Hamiltonian cycle $(\beta, P_1, \gamma, \gamma(0), R_1, \delta(0), R_2, \beta(0), \beta)$ can be constructed in $Q_4 - R$ (see Fig. 5(c)).

![Fig. 4](image1.png) An illustration for Lemma 4 with $n = 4$.

![Fig. 5](image2.png) Illustrations for Lemma 5.
Case 2: \( f(Q_3^4) = 1, f(D_0) = 3 \), and \( f(Q_3^5) = 0 \) or \( f(Q_3^5) = 2, f(D_0) = 2 \), and \( f(Q_3^5) = 0 \).
If \( f(Q_3^5) = 1 \), then there exists a fault-free Hamiltonian cycle \( C \) in \( Q_3^4 \) by Lemma 1. Since \( f(D_0) = 3 \), there exists an r-edge in \( C \). By Lemma 4, we can construct a fault-free Hamiltonian cycle in \( Q_4 \). Now we consider the case where \( f(Q_3^5) = 2, f(D_0) = 2 \), and \( f(Q_3^5) = 0 \). Let \( f_0 \) and \( f_1 \) be in \( D_0 \) and \( f_2 \) and \( f_3 \) in \( Q_3^4 \). There are two subcases to consider.

Subcase 2.1: There exists a faulty r-edge in \( Q_3^4 \).
Assume that \( f_2 \) is the faulty r-edge in \( Q_3^4 \) and, by Lemma 1, \( C \) is a Hamiltonian cycle in \( Q_3^4 \). Since \( f(D_0) = 2 \), there are at least four r-edges in \( C \). By Lemma 4, a Hamiltonian cycle can be constructed in \( Q_4 \). This concludes the proof of this lemma.

Subcase 2.2: None of \( f_2 \) and \( f_3 \) is a faulty r-edge.

By Lemma 1, there is a Hamiltonian cycle \( C \) in \( Q_3^4 \). Since \( f(D_0) = 2 \), there are at least four r-edges in \( C \). If \( C \) does not contain \( f_2 \), then, by Lemma 4, a fault-free Hamiltonian cycle can be constructed in \( Q_4 \). Assume that \( C \) contains \( f_2 = d_1(x) \) is a faulty edge. Then, by a similar argument as in Subcase 1.2, a Hamiltonian cycle \( \langle \beta, P_1, x^{(3)}, \alpha, P_2, \rho, i^{(6)}, R, \beta^{(6)}, \beta \rangle \) can be constructed in \( Q_4 \) if \( d_0(\rho) \) is a nonfaulty edge (see Fig. 5(a)). Note that, if \( d_0(\beta) \) is a faulty edge, then both \( d_3(\beta) \) and \( d_2(\beta) \) are nonfaulty edges. Since \( f(D_0) = 2 \), this implies that \( f_3 \) is a faulty r-edge, a contradiction. For the case where \( d_0(\rho) \) is a faulty edge, by assumption, \( \rho \) must be incident with \( f_1 \) and \( f_3 \). Since \( Q_4 \) is a bipartite graph and there is no f-cycle, there are at least five vertices in \( P_2 \). Let \( x \) and \( y \) be the other adjacent vertices of \( x^{(3)} \) and \( x \), respectively, in \( C \), and \( z \) be the other adjacent vertex of \( y \) in \( C \) (see Fig. 5(d)). By Theorem 3, a Hamiltonian cycle \( \langle y, y^{(0)}, R, x^{(0)}, x, P_1, \beta, \rho^{(6)}, R, \beta^{(6)}, \beta \rangle \) can be constructed in \( Q_4 \). This concludes the proof of this lemma.

Lemma 6. Let \( F \) be a faulty edge set in \( Q_3 \) with \( |F| \leq 7 \). If Constraints (1) and (2) are satisfied, then there is a Hamiltonian cycle in \( Q_5 \).

Proof. Clearly, we only need to consider the case where \( |F| = 7 \). There are five dimensional edge sets \( D_0, D_1, \ldots, D_5 \) in \( Q_5 \). We may assume without loss of generality that \( f(D_0) = 7 \). Since \( |F| = 7 \), by the pigeonhole principle, \( f(D_0) \geq 2 \). Thus, \( Q_2 \) can be decomposed into two subcubes \( Q_2^4 \) and \( Q_2^5 \) at dimension 0 with \( 0 \leq f(Q_2^4) \leq f(Q_2^5) \leq 5 \). Since \( f(D_0) = |F| = 7 \), the number of faulty edges in each dimension is less than 4 in \( Q_2^4 \). Since \( f(Q_2^4) \leq 5 \), there are at most two pivot vertices in \( Q_2^4 \). If \( 4 \leq f(Q_2^4) \leq 5 \), then there might be an f-cycle in \( Q_2^5 \). Notice that there is an f-cycle in \( Q_2^4 \), then there is no pivot vertex in \( Q_2^4 \). Otherwise, \( f(Q_2^4) > 5 \). Thus there are four cases to consider.

Case 1: There are exactly two pivot vertices in \( Q_2^4 \).
Let \( x \) and \( \beta \) be the pivot vertices in \( Q_2^4 \). Since each pivot vertex adjacent with three faulty edges in \( Q_2^2 \) and \( f(Q_2^4) \leq 5 \), \( x \) and \( \beta \) must be adjacent and edge \( xy \) is a faulty edge. This implies that \( f(Q_2^4) \geq 5 \). Since \( f(Q_2^4) = 0 \) and \( xy \) is a r-edge. Let \( f_0, f_1, \ldots, f_4 \) be the faulty edges in \( Q_2^4 \) and \( f_0 = xy \). After setting \( F = \{ f_0, f_1, f_2 \} \), it is easy to verify that Constraints (1) and (2) are satisfied in \( Q_2^4 \). By Lemma 5, there exists a Hamiltonian cycle \( C \) in \( Q_4 \). Note that \( f_2 \) and \( \beta \) be the other adjacent vertex of \( y \) in \( C \) (see Fig. 4 with \( n = 5 \)).

Case 2: There is exactly one pivot vertex in \( Q_2^4 \).
Let \( x \) be the pivot vertex in \( Q_2^4 \). Assume without loss of generality that \( d_1(x), d_2(x), \) and \( d_3(x) \) are faulty edges. There are two subcases to consider.

Subcase 2.1: \( d_0(x^{(1)}), d_0(x^{(2)}) \), and \( d_0(x^{(3)}) \) are faulty edges (see Fig. 6).
In this case, we need to consider that the faulty edge \( f \) is in \( Q_2^4 \). Recall that \( f(Q_2^4) \geq 4, 1 \leq f(D_1) \leq 2 \). If \( f \) is a faulty edge in \( Q_2^4 \), then \( f \) is a faulty edge in \( Q_2^4 \). Therefore, a Hamiltonian cycle \( \langle x, y, x^{(1)} \rangle \) can be constructed in \( Q_5 \).

Subcase 2.2: one of \( d_0(x^{(1)}), 1 \leq i \leq 3 \), is a nonfaulty edge.
Assume that \( d_0(x^{(1)}) \) is a nonfaulty edge. Let \( F = (F \setminus \{ f_1(x) \}) \cup E(Q_2^4) \). If \( Q_2^4 \) contains an f-cycle with respect to \( F \), then all 7 faulty edges in \( Q_2^4 \) are addressed, i.e., \( d_1(x), d_2(x), d_3(x), d_2^{(0)}(x), d_3^{(0)}(x), d_2^{(4)}(x), d_3^{(4)}(x) \). Thus, we can find a Hamiltonian path \( F \) starting from \( x \) and ending at \( x^{(4)} \). By Theorem 2, it follows directly that there is a fault-free Hamiltonian path \( \langle x^{(1)}, \beta, x^{(0)}, \beta \rangle \) in \( Q_2^4 \). Therefore, a Hamiltonian cycle \( \langle x, P, x^{(0)}, \beta, x^{(0)}, \beta \rangle \) can be constructed in \( Q_5 \).

Case 3: There exists an f-cycle in \( Q_3^4 \).
Assume that $\alpha$ and $\beta$ are the two vertices of degree 2 in the $f$-cycle of $Q_n^f$. Since $Q_n$ has no $f$-cycle, one of $d_0(\alpha)$ and $d_0(\beta)$ must be a nonfaulty edge. Assume without loss of generality that $d_0(\alpha)$ is a nonfaulty edge, and $d_1(\alpha)$ and $d_2(\alpha)$ are faulty edges. If one of $d_0(\alpha^{(1)})$ and $d_0(\alpha^{(2)})$ is not a faulty edge, say $d_0(\alpha^{(1)})$, then let $F = (F \setminus \{d_1(\alpha)\}) \cap E(Q_n^f)$. Since there are at least two faulty edges incident with $\beta$ and $f(D_0) \geq 2$ in $Q_n$, it is easy to check that $Q_n^f$ with $F$ satisfies all the requirements of Lemma 5 (see Fig. 8(a)). Thus there is a fault-free Hamiltonian cycle $C$ in $Q_n^f$ and $d_1(\alpha)$ is an edge in $C$. Since $f(D_0) \geq 2$ and $f(Q_n^f) \geq 4$, $f(Q_n^f) \leq 1$. By Theorem 2, it follows directly that there is a Hamiltonian cycle $(\alpha^{(0)}, \beta^{(0)}), \alpha^{(1)}, \alpha^{(0)}$ in $Q_n^f$. Thus a Hamiltonian cycle $(\alpha, \alpha^{(0)}, \beta, \alpha^{(1)}, \alpha^{(0)}, \beta, \alpha^{(1)}, \alpha^{(0)}, \beta)$ can be constructed in $Q_n - F$.

For the case where both of $d_0(\alpha^{(1)})$ and $d_0(\alpha^{(2)})$ are faulty edges, there exists a faulty edge $f_1 = d_0(\beta)$, for $h = 1, 2$, such that $d_0(\beta^{(0)})$ is a nonfaulty edge. Assume without loss of generality that $f_1 = d_1(\beta)$ and let $F = (F \setminus \{f_1\}) \cap E(Q_n^f)$. It is easy to check that $Q_n^f - F$ satisfies the requirements of Lemma 5. Thus there exists a Hamiltonian cycle $C$ containing $f_1$ in $Q_n^f$ (see Fig. 8(b)). By Theorem 2, we can find a fault-free Hamiltonian path $(\alpha^{(0)}, \beta, \beta^{(1)}, \beta^{(0)}, \alpha^{(0)})$ in $Q_n^f$. Therefore, a fault-free Hamiltonian cycle $(\alpha, \alpha^{(0)}, \beta, \beta^{(1)}, \beta^{(0)}, \beta, \alpha^{(0)}, \alpha)$ can be constructed in $Q_n$ (see Fig. 8(c)).

**Case 4:** Neither a pivot vertex nor an $f$-cycle is in $Q_n^f$.

Since $f(D_0) = \max\{f(D_0), f(D_1), \ldots, f(D_n)\}, f(Q_n^f) \leq 5$ and $f(Q_n^f) \leq 2$. If $f(Q_n^f) \leq 4$, then, by Lemma 5 and Theorem 2, there exists a Hamiltonian cycle $C$ with an $r$-edge, say $\alpha\beta$, in $Q_n^f$ and there is a fault-free Hamiltonian path $(\alpha^{(0)}, \beta, \beta^{(0)}, \alpha^{(0)})$ in $Q_n^f$. Consequently, a Hamiltonian cycle $(\alpha, \alpha^{(0)}, \beta, \beta^{(0)}, \beta, \beta^{(0)}, \alpha^{(0)}, \alpha^{(0)})$ can be constructed in $Q_n - F$ (see Fig. 4 by replacing $Q_n$, $Q_n^{L-1}$, and $Q_n^{R-1}$ with $Q_n^f$, $Q_n^L$, and $Q_n^R$, respectively). If $f(Q_n^f) = 5$, then there exists at least one faulty $r$-edge, say $e = \alpha\beta$, in $Q_n^f$. Let $F = (F \setminus \{e\}) \cap E(Q_n^f)$. Thus $Q_n^f$ with $F$ satisfies all the requirements of Lemma 5, and there exists a Hamiltonian cycle $C$ with at most one faulty edge in $Q_n^f$. Note that if there is a faulty edge in $C$, then it must be $e$. By using a similar construction as the case for $f(Q_n^f) \leq 4$, a Hamiltonian cycle can be constructed in $Q_n - F$. This completes the proof. \qed

**Theorem 7.** Let $F$ be a faulty edge set in $Q_n$ with $n \geq 5$ and $|F| \leq 3n - 8$. If Constraints (1) and (2) are satisfied, then there is a Hamiltonian cycle in $Q_n - F$.

**Proof.** We prove this theorem by induction on $n$. The basis, i.e., $n = 5$, is proved in Lemma 6. In the inductive step, we show that the statement also holds for $n > 5$. Let $F$ be a set of faulty edges with $|F| \leq 3n - 8$.

First, find a dimension $i$ with $f(D_i) \geq 1$ such that there is no pivot vertex in $Q_{n-1}^L$ and $Q_{n-1}^R$, if exists, after $Q_n$ is decomposed at dimension $i$. We can find such a dimension as follows. If there is no vertex incident with $n - 2$ faulty edges, then arbitrarily choose a dimension in which a faulty edge lies as dimension $i$; otherwise, choose a dimension, if it exists, in
which no g-edge lies. Since |F| ≤ 3n − 8 in Qₙ, there are at most three vertices such that each of them is incident with n − 2 faulty edges in Qₙ. This implies that at most six dimensions cannot be chosen as dimension i. Thus we can always find such a dimension except when n ≤ 6. Accordingly, in the following, we shall show that there exists a fault-free Hamiltonian cycle in Qₙ in which there are exactly three vertices incident with n − 2 faulty edges. Besides, we shall also show that there exists a fault-free Hamiltonian cycle in Qₙ for n ≥ 6 when such a dimension exists. For simplicity, we may assume without loss of generality that dimension 0 is such a dimension.

Now we consider the case where there exist three vertices, say a, b, and c, such that each of them is incident with 4 faulty edges in Qₙ. Let dᵢₐ(a), dᵢₜ(a), dᵢₜ(b), dᵢₜ(c), and dᵢₜ(c) be their g-edges with iₜ ≠ iᵢ if i ≣ j for 1 ≤ i,j ≤ 6. Furthermore, vertex b is adjacent to a and c. We choose iₜ as dimension 0. Thus b is a pivot vertex, f(D₀) = 2, and f(Qₙ) = 8. Since all faulty edges |F| = 3 × 6 − 8 = 10 are addressed, one of the faulty edges incident with b, say dᵢₜ(b), is a faulty r-edge. Let F = F \ {dᵢₜ(b)} ∩ E(Qₙ). Note that |F| = 7. By the induction hypothesis, there exists a Hamiltonian cycle C containing dᵢₜ(b) in Qₙ − F. By Theorem 2, there is a fault-free Hamiltonian path (b(0), R, b(6)) in Qₙ. Consequently, a Hamiltonian cycle (b(4), c − bb(4), b, b(0), R, b(6), b(4)) can be constructed in Qₙ − F (see Fig. 9). Thus this theorem holds for this case.

Now we show that there exists a fault-free Hamiltonian cycle in Qₙ in which f(D₀) ≥ 1 and there is no pivot vertex. We claim that if there is an f-cycle in Qₙ, then it must be in Qₙ − 1. If there exists an f-cycle in Qₙ, then there are at least 2n − 4 faulty edges in Qₙ. Since at least one faulty edge is in dimension 0, 0 < f(Qₙ) ≤ f(Qₙ − 1) ≤ 3n − 9. This implies that f(Qₙ − 1) ≤ 3n − 9 < n(2n − 1) − 4 for n ≥ 5. Thus the claim holds and we only need to consider whether Qₙ − 1 contains an f-cycle or not. There are four cases to consider.

Case 1: f(D₀) = 1, f(Qₙ) = 0, and there exists an f-cycle in Qₙ − 1 − F.

Let f₀ be the faulty edge in dimension 0 of Qₙ, and a and b be the two vertices of degree 2 in the f-cycle. Since there is no f-cycle in Qₙ − 1 − F, one of d₀(a) and d₀(b) must be a nonfaulty edge. We may assume without loss of generality that d₀(a) is a nonfaulty edge and d₁(a) = d₄(a), d₅(a) = d₆(a), and dₐ(a) = dₐ(a) are faulty edges. Since f(D₀) = 1, there exists a faulty edge f₁ = dₐ(a), for some 1 ≤ h ≤ n − 3, such that dₐ(a) is a nonfaulty edge. Clearly, there exists another faulty edge f₁ = ab in Qₙ − 1 which is nonadjacent to f₁ and both d₀(a) and d₀(b) are nonfaulty edges (see Fig. 10(a)). Let F = F \ {f₀, f₁}. It is easy to check that Constraints (1) and (2) are satisfied in Qₙ − 1 − F. Furthermore, in Qₙ − 1 − F, f(Qₙ) = f(Qₙ − 1) − 3 = 3n − 8 ≤ 3n − 11 = 3(n − 1) − 8. This implies that Qₙ − 1 − F satisfies the requirement of this theorem. By the induction hypothesis, there exists a Hamiltonian cycle C in Qₙ − 1 − F which contains f₁. If C also contains f₂, then we can divide C into four subpaths: (x, P₁, x(0), R₁, y(0),) and (x(0), R₁, x(0), (b, y)). By Theorem 3, there exist two vertex-disjoint paths (x(0), R₁, y(0)) and (x(0), R₂, x(0), (b, y)) in Qₙ − 1 spanning V(Qₙ − 1). Therefore, a Hamiltonian cycle (x, P₂, y, y(0), R₁, x(0), x, P₁, x(0), x(0), R₂, x(0), x) can be constructed in Qₙ − F (see Fig. 10(a)). For the case where C does not contain f₂, by Theorem 2, we can find a Hamiltonian path (x(0), R₁, x(0), (b, y)) spanning all vertices in Qₙ − 1 (see Fig. 10(b)). As a consequence, a Hamiltonian cycle (x, C − f₁, x(0), R₁, x(0), x) can be constructed in Qₙ − F.
be the faulty edge in dimension 0 of $\in C_0$ with $\in Q_{n-1}$, and $a$ is addressed, i.e., the faulty edges $1$ and $/C_0$ are in $\in Q_n$ (see Fig. 8 (a) by replacing $hypothesis, there exists a Hamiltonian cycle $/C_0$ are defined in Case 1. Then by using a similar argument as in Case 1, a Hamiltonian cycle $/C_0$ can be found in $Q_{n-1}$ with $/C_0$ is in $Q_n - F$ (see Fig. 4). If both of $f_1$ and $f_2$ are in $C$, then similar to Subcase 1.3 of Lemma 5, a Hamiltonian cycle can be constructed in $Q_n - F$ (see Fig. 5(b) by replacing $Q_4, Q_5$, and $Q_6$ with $Q_n, Q_{n-1}$, and $Q_{n-2}$).

Case 2: $f(D_0) = 1, f(Q_{n-1}^R) = 0$, and there does not exist an $f$-cycle in $Q_{n-1}^L$.

Let $f_0$ be the faulty edge in dimension 0 of $Q_n$, and $f_1 = x\beta$ and $f_2 = y\gamma$ are two nonadjacent faulty edges in $Q_{n-1}$ in which both $d_0(\alpha)$ and $d_0(\beta)$ (respectively, $d_0(\gamma)$ and $d_0(\delta)$) are fault-free. Let $F = F - \{f_0, f_1, f_2\}$. By using a similar argument as in Case 1, a Hamiltonian cycle $C$ can be found in $Q_{n-1}^L$ with $F$ in which $C$ contains at least one of $f_1$ and $f_2$. If none of $f_1$ and $f_2$ is in $C$, then by an analogous reasoning as in Lemma 4, a Hamiltonian cycle can be constructed in $Q_n - F$ (see Fig. 4). If both of $f_1$ and $f_2$ are in $C$, then similar to Subcase 1.3 of Lemma 5, a Hamiltonian cycle can be constructed in $Q_n - F$ (see Fig. 5(b) by replacing $Q_4, Q_5$, and $Q_6$ with $Q_n, Q_{n-1}$, and $Q_{n-2}$).

Case 3: $f(Q_{n-1}^L) \leq 3n - 10, f(D_0) \geq 1$, and there exists an $f$-cycle in $Q_{n-1}^L$.

Let $F = F - \{f_0, f_1\}$ where $f_0$ and $f_1$ are defined in Case 1. Then by using a similar argument as in Case 1, there exists a Hamiltonian cycle $C$ containing $f_1$ in $Q_{n-1}^L$. Since $f(Q_{n-1}^L) \leq \frac{1}{2}(3n - 9) < 2(n - 1) - 5$ for $n > 5$, by Theorem 2, we can find a Hamiltonian path $(x^{(h,0)}, R, x^{(h,0)})$ in $Q_{n-1}^L$. Therefore, a fault-free Hamiltonian cycle $(x, C - f_1, x^{(h,0)}, R, x^{(h,0)}, x)$ can be constructed in $Q_n$ (see Fig. 8(a) by replacing $Q_5, Q_6, Q_4, x^{(1)}$, and $x^{(1)}$ with $Q_n, Q_{n-1}, Q_{n-2}$, $x^{(h,0)}$, and $x^{(h,0)}$, respectively).

If $d_0(\alpha^{(1)}), d_0(\alpha^{(2)}), \ldots, d_0(\alpha^{(n-3)})$ are all faulty edges, then all $3n - 8$ faulty edges in $Q_n$ are addressed, i.e., the faulty edges adjacent with $x$ and $\beta, f_0$, and $d_0(\alpha^{(h)})$ for $1 \leq i \leq n - 3$. Thus it remains to consider that there exists a faulty edge $f_3 = d_0(\beta)$ for some $1 < h \leq n - 3$ such that $d_0(\beta^{(h)})$ is a nonfaulty edge. We may assume without loss of generality that $f_3 = d_0(\beta)$ and let $F = (F - \{f_3\}) \cap E(Q_{n-1}^L)$. It is easy to check that $Q_{n-1}^L - F$ satisfies the requirements of this theorem. Thus, by the induction hypothesis, there exists a Hamiltonian cycle $C$ containing $f_3$ in $Q_{n-1}^L$ (see Fig. 10(c)). By Theorem 2, we can find a fault-free Hamiltonian path $(x^{(0)}, R, \beta^{(1)})$ in $Q_{n-1}^R$. Therefore, a fault-free Hamiltonian cycle $(x, x^{(n-1)}, \beta, x^{(n-2)}, C - \{x, x^{(n-1)}, x^{(n-2)}, \beta\}, \beta^{(1)}, \beta^{(1)}, R, x^{(0)}, x)$ can be constructed in $Q_n$ (see Fig. 10(d)).
Case 4: If \( f(Q^k_{n+1}) \leq 3n - 10 \), then \( f(D_0) \geq 1 \), and there is no \( f \)-cycle in \( Q^k_{n+1} \).
If \( f(Q^k_{n+1}) = 3n - 10 \), then let \( F' = F - \{f_0, f_1\} \cap E(Q^k_{n+1}) \); otherwise, let \( F' = F \cap E(Q^k_{n+1}) \), where \( f_0 \) is the faulty edge in dimension 0 of \( Q_n \), and \( f_1 = \alpha \beta \) is a faulty \( r \)-edge in \( Q^k_{n+1} \).
It is easy to check that \( Q^k_{n+1} \) with \( F' \) satisfies the requirements of this theorem. Thus, by the induction hypothesis, there exists a Hamiltonian cycle \( C \) in \( Q^k_{n+1} \). Note that \( C \) might contain \( f_1 \). Since \( f(Q^k_{n+1}) \leq \frac{1}{2}(3n - 9) < 2(n - 1) - 5 \) for \( n > 5 \), by Theorem 2, we can find a fault-free Hamiltonian path \( \langle \alpha^{(0)}, R, \beta^{(0)} \rangle \) in \( Q^k_{n+1} \). Therefore, a fault-free Hamiltonian cycle \( \langle \alpha, C - f_1, \beta, \beta^{(0)}, R, \alpha^{(0)}, \alpha \rangle \) can be constructed in \( Q_n \). (see Fig. 4 with \( i = 0 \)). If \( f_1 \) is not in \( C \) or \( f(Q^k_{n+1}) < 3n - 10 \), then choose any \( r \)-edge in \( C \) as the edge \( \alpha \beta \). Similarly, a fault-free Hamiltonian cycle can be constructed in \( Q_n \). This completes the proof.

4. Conclusion

Finally, we want to point out that our result is optimal under Constraints (1) and (2). The hypercube \( Q_n \) depicted in Fig. 11 has \( 3n - 7 \) faulty edges in which vertices \( a, c, \) and \( e \) are incident with \( n - 2, n - 2, \) and \( n - 3 \) faulty edges, respectively. We can find that there is no Hamiltonian cycle in \( Q_n \) if it has a subgraph isomorphic to the graph as shown in Fig. 11. This reveals that if there is no constraint, then the maximal number of faulty edges is \( n - 2 \) so that there still exists a fault-free Hamiltonian cycle in \( Q_n \). If \( Q_n \) satisfies Constraint (1), then the number of maximal allowable faulty edges is \( 2n - 5 \). If \( Q_n \) satisfies Constraints (1) and (2), then the number of maximal allowable faulty edges becomes \( 3n - 8 \). Thus it is interesting to find out forbidden graphs such that there still exists a Hamiltonian cycle in \( Q_n \). In particular, what is the number of maximal faulty edges in \( Q_n \) under the constraint that no forbidden graph exists?

Acknowledgment

The authors would like to thank anonymous referees for their careful reading of corrections and useful comments which helped to improve the paper.

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