SUMMARY A queue layout of a graph $G$ consists of a linear order of its vertices, and a partition of its edges into queues, such that no two edges in the same queue are nested. The queue number $qn(G)$ is the minimum number of queues required in a queue layout of $G$. The Cartesian product of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \times G_2$, is the graph with $((v_1, v_2) : v_1 \in V_1$ and $v_2 \in V_2$) as its vertex set and an edge $((v_1, u_2), (u_1, v_2))$ belongs to $G_1 \times G_2$ if and only if either $(u_1, v_1) \in E_1$ and $v_2 = u_2$ or $(u_1, v_1) \in E_2$ and $u_2 = v_2$. Let $T_{k_1, k_2, \ldots, k_n}$ denote the $n$-dimensional toroidal grid defined by the Cartesian product of $n$ cycles with varied lengths, i.e., $T_{k_1, k_2, \ldots, k_n} = C_{k_1} \times C_{k_2} \times \cdots \times C_{k_n}.$ Let $Tk$ be the $k$-ary $n$-cube as a subgraph of $T_{k_1, k_2, \ldots, k_n}$, where $k_i$ is a cycle of length $k_i \geq 3$. If $k_i = k_2 = \cdots = k_n = k$, the graph is also called the $k$-ary $n$-cube and is denoted by $Q_k^n$. In this paper, we deal with queue layouts of toroidal grids and show the following bound: $qn(T_{k_1, k_2, \ldots, k_n}) \leq 2n - 2$ if $n \geq 2$ and $k_i \geq 3$ for all $i = 1, 2, \ldots, n$. In particular, for $n = 2$ and $k_1, k_2 \geq 3$, we acquire $qn(T_{k_1, k_2}) = 2$. Recently, Pai et al. (Inform. Process. Lett. 110 (2009) pp.50–56) showed that $qn(Q_k^n) \leq 2n - 1$ if $n \geq 1$ and $k \geq 9$. Thus, our result improves the bound of $qn(Q_k^n)$ when $n \geq 2$ and $k \geq 9$.

**Key words:** Queue layout, Toroidal grids, Cartesian product, Arched leveled-planar graphs

1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A vertex ordering $\sigma$ of $G$ is a bijection from $V(G)$ to $\{1, 2, \ldots, |V(G)|\}$. For $u, v \in V(G)$, we write $u <_\sigma v$ if $\sigma(u) < \sigma(v)$. A $k$-queue layout of a graph $G$ consists of a vertex ordering $\sigma$ and a partition of its edges into $k$ queues such that no two edges in the same queue are nested (i.e., two edges $(u, v), (x, y) \in E(G)$ are nested if $u <_\sigma x <_\sigma y <_\sigma v$ or $x <_\sigma u <_\sigma v <_\sigma y$). The queue number of a graph $G$, denoted by $qn(G)$, is the minimum $k$ such that $G$ has a $k$-queue layout. A graph $G$ is a $k$-queue graph if $qn(G) \leq k$.

Queue layouts were first introduced by Heath et al. [13], [16]. There are many applications of queue layouts in computer science, including sorting permutations [7], [18], [22], [28], [32], parallel process scheduling [1], matrix computations [27] and graph drawing [5], [9], [33]. In particular, queue layouts of interconnection networks have applications to the Diogenes approach to testable fault-tolerant arrays of processors [30]. Heath and Rosenberg [16] showed that recognizing a $k$-queue graph is NP-complete even if $k = 1$. Thus, further investigations tended to study bounds on queue number for certain families of graphs [5], [6], [8], [11]–[16], [24]–[26], [29], [33], [34].

This paper deals with queue layouts of a family of graphs called toroidal grids. Let $k_i \geq 3$ be integers for $i = 1, 2, \ldots, n$. The $n$-dimensional toroidal grid, denoted by $T_{k_1, k_2, \ldots, k_n}$, is a graph consisting of $N = k_1 \times k_2 \times \cdots \times k_n$ vertices, each of which is associated with a label $x = \langle x_1, x_2, \ldots, x_n \rangle$ where $x_i \in \{0, 1, \ldots, k_i - 1\}$ for $i = 1, 2, \ldots, n$, and two vertices $x = \langle x_1, x_2, \ldots, x_n \rangle$ and $y = \langle y_1, y_2, \ldots, y_n \rangle$ are adjacent if and only if there exists an integer $j \in \{1, 2, \ldots, n\}$ such that $x_j \equiv y_j \pm 1 \pmod{k_j}$ and $x_i = y_i$ for all $i \in \{1, 2, \ldots, n\} \setminus \{j\}$. For example, Fig. 1(a), 1(b) and 1(c) respectively depict $T_3$, $T_{3,3}$ and $T_{3,3,3}$, where a vertex $\langle x_1, x_2, \ldots, x_n \rangle$ is written as $x_1 x_2 \cdots x_n$ for notational convenience.

The class of $T_{k_1, k_2, \ldots, k_n}$ includes $k$-ary $n$-cubes as a subgraph of $T_{k_1, k_2, \ldots, k_n}$. In particular, queue layouts of interconnection networks have applications to the Diogenes approach to testable fault-tolerant arrays of processors [30]. Heath and Rosenberg [16] showed that recognizing a $k$-queue graph is NP-complete even if $k = 1$. Thus, further investigations tended to study bounds on queue number for certain families of graphs [5], [6], [8], [11]–[16], [24]–[26], [29], [33], [34].

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class. The \( k \)-ary \( n \)-cube \( Q_n^k \) is defined as \( T_{k_1,k_2,...,k_n} \) with the restriction \( k_1 = k_2 = \cdots = k_n = k \geq 3 \). In particular, the class of \( Q_n^k \) is called the ternary \( n \)-cubes. The \( n \)-dimensional hypercube \( Q_n \) is a graph with \( 2^n \) vertices such that each vertex is associated with a label \( x = (x_1, x_2, \ldots, x_n) \) where \( x_i \in \{0, 1\} \) and two vertices are linked by an edge if and only if they differ in exactly one coordinate. We note that most current supercomputers (e.g., Cray T3D [23] and Cray T3E [31], iWARP [3], Intel Teraflops system [4], Fujitsu AP1000 [17], and Chaos system [2], [20]) are based on the 2- and 3-dimensional toroidal grids for low-latency and high-bandwidth inter-processor communication. Also, \( Q_n \) has resulted in several experimental and commercial machines including Ncube 6400 [21] and Intel iPSC [19].

In this paper, we establish an upper bound on \( qn(T_{k_1,k_2,...,k_n}) \) by using induction. For the base step, we provide a construction scheme to produce a 2-queue layout of \( T_{k_1,k_2} \) for \( k_1, k_2 \geq 3 \). In the following table, we summarize some recent results of queue layouts for the aforementioned classes of graphs.

### 2. Arched Leveled-Planar Graphs

Because our construction scheme relies on a vertex ordering yielded from a particular planar embedding introduced by Heath and Rosenberg [16], we now present the concept of such planar embeddings. A graph \( G = (V, E) \) is **levelled-planar** if \( V \) can be partitioned into subsets \( V_1, V_2, \ldots, V_m \) so that \( G \) has a planar embedding in which all vertices of \( V_i \) are on a vertical line \( \ell_i \) defined by \( \ell_i = \{(x, y) | y \in \mathbb{R} \} \), and each edge in \( E \) is embedded as a straight-line segment wholly between \( \ell_i \) and \( \ell_{i+1} \) for some \( i \). Such a planar embedding is called a **levelled-planar embedding** [16]. A levelled-planar graph under this embedding has a natural ordering on vertices, called the **induced order**, by scanning line \( \ell_i \) from bottom to top for each \( i = 1, 2, \ldots, m \) and labeling the vertices 1, 2, \ldots, |\( V \)| as they are encountered. Heath and Rosenberg [16] further showed that every levelled-planar graph is a 1-queue graph and the induced order of its vertices yields a 1-queue layout.

To completely characterize the class of 1-queue graphs, Heath and Rosenberg [16] introduced a wider class of graphs called arched levelled-planar graphs, which contains levelled-planar graphs as a subclass. Consider a levelled-planar graph \( G \). For \( 1 \leq i \leq m \), let \( b_i \) be the first vertex (i.e., the bottom vertex) and \( t_i \) be the last vertex (i.e., the top vertex) in line \( \ell_i \). Let \( s_i \) be the first vertex in \( \ell_i \), which is adjacent to some vertex in \( \ell_{i+1} \), or, let \( s_i = t_i \) if there are no edges between lines \( \ell_i \) and \( \ell_{i+1} \). An **arched levelled-planar graph** is an augmentation of \( G \) that adds some new edges (zero or more) to \( G \) by connecting vertex \( t_i \) with vertex \( j_i \), where \( b_i \leq j_i \leq \min\{t_{i-1}, s_i\} \), and each added edge is called an **arch** of \( G \). An arched levelled-planar graph can be embedded in the plane by drawing the arches around \( \ell_i \) such that arches do not cross each other. Under this embedding, the induced order of an arched levelled-planar graph is the same as that of the corresponding levelled-planar graph without arches. The edges that are not arcs are called **arched edges**. An arched levelled-planar graph is **maximal** if it cannot be augmented with further arches or levelled edges. A maximal arched levelled-planar graph on \( n \) vertices contains exactly \( 2n - 3 \) edges [16]. This is useful for establishing lower bounds on queue number. For example, Fig. 2(a) shows a levelled-planar graph, Fig. 2(b) shows an arched levelled-planar graph with arches (4, 5) and (6, 8), and Fig. 2(c) shows a maximal arched levelled-planar graph.

Our construction scheme of queue layout is based on the following known results.

**Theorem 1** (Heath and Rosenberg [16]): A graph \( G \) is a 1-queue graph if and only if \( G \) is an arched levelled-planar graph. In particular, the induced order of vertices in an arched levelled-planar graph yields a 1-queue layout.

**Corollary 2:** A graph \( G \) is a \( k \)-queue graph if and only if \( G \) can be partitioned into \( k \) edge-disjoint spanning subgraphs such that all subgraphs have arched levelled-planar embeddings with the same induced order.

Also, from Theorem 1 and the concept of maximal arched levelled-planar graphs, we immediately obtain the following lower bound on queue number for general graphs.

**Corollary 3:** For any graph \( G \) with \( |V(G)| > 2 \), the maximum number of edges that can be assigned to a single queue is \( 2|V(G)| - 3 \). That is, \( qn(G) \geq \lceil \frac{|E(G)|}{2|V(G)|-3} \rceil \).

### 3. Queue Layout of \( T_{k_1,k_2} \)

The **Cartesian product** of two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), denoted by \( G_1 \times G_2 \), is the graph with \( \{(v_1, v_2) \in V_1 \times V_2 | (v_1, v_2) \in E_1 \times G_2 \} \) as its vertex set and an edge \( \{(u_1, u_2), (v_1, v_2) \} \) belongs to \( G_1 \times G_2 \) if and only if either \( (u_1, v_1) \in E_1 \) or \( u_1 = v_1 \) and \( (u_2, v_2) \in E_2 \). The **m-th power** of a graph \( G = (V, E) \) is a graph with the same vertex as \( G \), denoted by \( G^m = (V, E^m) \), such that \((u, v) \in E^m\) if and only if there is a path with length no more than \( m \) between \( u \) and \( v \) in \( G \). A **generalized n-dimensional toroidal grid** is defined by \( C_m^{k_1} \times C_m^{k_2} \times \cdots \times C_m^{k_i} \), where \( C_m^{k_i} \) is the m-th power of a cycle of length \( k_i \geq 2m + 1 \). In [34], Wood studied queue layouts of graph products and graph powers, and
showed the following bounds on queuenumber of a generalized toroidal grid.

**Theorem 4** (Wood [34]): The queuenumber of a graph $G = C_{k_1}^m \times C_{k_2}^m \times \cdots \times C_{k_n}^m$ satisfies

$$\frac{nm}{2} < qn(G) \leq (2n-1)m.$$  

A graph is **unicyclic** if every connected component has at most one cycle. Heath and Rosenberg [16] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle is a 1-queue graph. Since $T_k$ is isomorphic to a cycle of length $k$, $qn(T_k) = 1$ for all $k \geq 3$. Also, by Theorem 4, $qn(T_{k_1,k_2})$ is either 2 or 3 for any $k_1, k_2 \geq 3$. In the following three lemmas, we provide construction schemes to determine the queuenumber of $T_{k_1,k_2}$. Without loss of generality we consider $k_2 \geq k_1 \geq 3$.

**Lemma 5:** $qn(T_{3,k}) = 2$ for $k \geq 3$.

**Proof:** Obviously, $T_{3,k}$ contains $3k$ vertices and $6k$ edges. To show that there exists a 2-queue layout of $T_{3,k}$, by Corollary 2, we need to partition $T_{3,k}$ into two edge-disjoint spanning subgraphs, say $G_1$ and $G_2$, such that both subgraphs have arched leveled-planar embeddings with the same induced order. For $G_1$, its arched leveled-planar embedding consists of $k + 1$ levels such that all vertices are arranged according to the following rules:

1. The vertex $\langle 2, j \rangle$ is located in the top of line $\ell_{j+2}$ for $j = 0, 1, \ldots, k-1$.
2. The vertex $\langle 0, j \rangle$ is located in the middle of line $\ell_{j+1}$ for $j = 0, 1, \ldots, k-1$.
3. The vertex $\langle 1, j \rangle$ is located in the bottom of line $\ell_{j+2}$ for $j = 0, 1, \ldots, k-1$.

Clearly, we have leveled edges $\langle (0, j), (0, j+1) \rangle$, $\langle (1, j), (1, j+1) \rangle$ and $\langle (2, j), (2, j+1) \rangle$ for $j = 0, 1, \ldots, k-2$, and $\langle (0, j), (1, j) \rangle$ and $\langle (0, j), (2, j) \rangle$ for $j = 0, 1, \ldots, k-1$. It is easy to check that no leveled edges cross each other. Also, we have arches $\langle (1, j), (2, j) \rangle$ for $j = 0, 1, \ldots, k-1$. As a result, $G_1$ has $6k - 3$ edges and is an arched leveled-planar graph (see Fig. 3(b) for such an embedding of $G_1$ for $T_{3,5}$).

Let $\sigma$ be the induced order of $G_1$ under the above leveled-planar embedding. Also, we know that $G_2$ contains only three edges $\langle (0, 0), (0, k-1) \rangle$, $\langle (1, 0), (1, k-1) \rangle$ and $\langle (2, 0), (2, k-1) \rangle$. To configure an arched leveled-planar embedding of $G_2$, we partition all vertices into two levels, where vertices from $(0,0)$ to $(1,k-2)$ in $\sigma$ are consecutively arranged in line $\ell_1$ and all remaining vertices are consecutively arranged in line $\ell_2$. Since $(0,0) <_\sigma (1,0) <_\sigma (2,0)$ and $(0,k-1) <_\sigma (1,k-1) <_\sigma (2,k-1)$, the three edges do not cross each other (see Fig. 3(b) for such an embedding of $G_2$ for $T_{3,5}$).

For ease of description, all edges contained in $T_{k_1,k_2}$ are divided into two types: **wrapped edges** and **mesh edges**. The former contains edges $\langle (i, 0), (i, k_i - 1) \rangle$ and $\langle (i, j), (k_i - 1, j) \rangle$ for $i = 0, 1, \ldots, k_1 - 1$ and $j = 0, 1, \ldots, k_2 - 1$, while the latter contains all non-wrapped edges. In particular, since our construction scheme partitions $T_{k_1,k_2}$ into two edge-disjoint spanning subgraphs, say $G_1$ and $G_2$, the mesh edges dealt out to $G_2$ are called **splitting edges**.
Lemma 6: \( qn(T_{2h+4,k}) = 2 \) for \( h \geq 0 \) and \( k \geq 2h + 4 \).

Proof: Obviously, \( T_{2h+4,k} \) contains \((2h + 4)k\) vertices and \(4(h + 2k)\) edges. By Corollary 2, we partition \( T_{2h+4,k} \) into two edge-disjoint spanning subgraphs \( G_1 \) and \( G_2 \), such that their leveled-planar embeddings have a common induced order. For \( G_1 \), its leveled-planar embedding consists of \( 2k + h \) levels. In each level, the indicated line prepares \( h + 3 \) positions for arranging vertices and all positions are ranked in a top-down fashion (i.e., the first position is on the top in the line and the last position is on the bottom in the line.) We now arrange vertices in two parallelgrams according to the following rules:

1. For the first parallelgram, the vertex \((i, j)\) is located in the \((i + 2)\)-th position of line \( \ell_{i+j+1} \) for \( i = 0, \ldots, h + 1 \) and \( j = 0, \ldots, k - 1 \).

2. For the second parallelgram, the vertex \((i, j)\) is located in the \((2h + 4 - i)\)-th position of line \( \ell_{i+2h+1+j} \) for \( i = h + 2, h + 3, \ldots, 2h + 3 \) and \( j = 0, \ldots, k - 1 \). Thus, we have the following leveled edges: \((i, j), (i, j + 1)\) for \( i = 0, 1, \ldots, 2h + 3 \) and \( j = 0, 1, \ldots, k - 1 \) are wrapped edges, and \((h + 1, j), (h + 2, j)\) for \( j = 0, 1, \ldots, k - 1 \) are splitting edges. Let \( \sigma \) be the induced order of \( G_1 \) under the above leveled-planar embedding. Then, we partition all vertices of \( G_2 \) into three levels, where vertices from \((0, 0)\) to \((1, k - 2)\) in \( \sigma \) are consecutively arranged in line \( \ell_1 \), vertices from \((0, k - 1)\) to \((2h + 2, k - 2)\) in \( \sigma \) are consecutively arranged in line \( \ell_2 \), and all remaining vertices are consecutively arranged in line \( \ell_3 \). Note that \((1, k - 2)\) (respectively, \((2h + 2, k - 2)\)) is the second rightmost vertex in the second topmost row of the first parallelgram (respectively, the second parallelgram). Hence, there are \( 2k \) leveled edges between lines \( \ell_1 \) and \( \ell_2 \) of three types, as follows:

(i) wrapped edges \((0, j), (2h + 3, j)\) for \( j = 0, 1, \ldots, k - 2 \),

(ii) wrapped edges \((0, 0), (i, k - 1)\) for \( i = 0, 1, \ldots, h + 1 \), and

(iii) splitting edges \((h + 1, j), (h + 2, j)\) for \( j = 0, 1, \ldots, k - 2 \).

We first observe that for two wrapped edges \((0, 0), (2h + 3, j)\) and \((0, j'), (2h + 3, j')\) where \( 0 \leq j < j' \leq k - 2 \), we have \((0, 0) <_{\sigma} (0, j') \) and \((2h + 3, j) <_{\sigma} (2h + 3, j')\). This shows that all type-\((i)\) edges do not cross each other. Moreover, for two type-\((ii)\) edges \((i, 0), (i, k - 1)\) and \((i + 1, 0), (i + 1, k - 1)\) where \( i = 0, 1, \ldots, h \), there is a type-\((i)\) edge \((0, i), (2h + 3, i)\) such that \( (0, 0) <_{\sigma} (0, i) \) (particularly, \( (0, 0) = (0, i) \) if \( i = 0 \)) and \( (i, k - 1) <_{\sigma} (2h + 3, i) \). Similarly, for two splitting edges \((h + 1, j), (h + 2, j)\) and \((h + 1, j + 1), (h + 2, j + 1)\) where \( j = 0, 1, \ldots, k - h - 3 \), there is a type-\((i)\) edge \((0, h + j + 1), (2h + 3, h + j + 1)\) such that \( (h + 1, j) <_{\sigma} (h + 1, j + 1) \) and \( (h + 2, j) <_{\sigma} (2h + 3, h + j + 1) \) (\( h + 1, j + 1 \) for \( h + 1, j + 1 \)). That is, if we consider edges between \( \ell_1 \) and \( \ell_2 \) in a bottom-up fashion, then type-\((i)\) edges and type-\((ii)\) edges (respectively, type-\((iii)\) edges) appear alternately, and all type-\((ii)\) edges turn up before type-\((iii)\) edges. Therefore, all edges between \( \ell_1 \) and \( \ell_2 \) do not cross each other.

We now consider the remaining \((2h + 4)\) edges between lines \( \ell_2 \) and \( \ell_3 \) of three types, as follows:

(i') wrapped edge \((0, k - 1), (2h + 3, k - 1)\);

(ii') wrapped edges \((i, 0), (i, k - 1)\) for \( i = h + 2, h + 3, \ldots, 2h + 3 \);

(iii') splitting edges \((h + 1, j), (h + 2, j)\) for \( j = k - h - 1, k - h, \ldots, k - 1 \).

It is clear that for two wrapped edges \((i, 0), (i, k - 1)\) and \((j', 0), (j', k - 1)\) where \( h + 2 \leq i < j' < 2h + 3 \), we have \((j', 0) <_{\sigma} (i, 0) \) and \((j', k - 1) <_{\sigma} (i, k - 1) \). Thus, all type-\((ii)\) edges do not cross each other. Moreover, if we consider edges between \( \ell_2 \) and \( \ell_3 \) in a bottom-up fashion, it is easy to see that \((0, k - 1), (2h + 3, k - 1)\) is the first edge, and type-\((ii')\) edges and type-\((iiii')\) edges appear alternately. Consequently, all edges between \( \ell_2 \) and \( \ell_3 \) do not cross each other (see Fig. 4(b) for such an embedding of \( G_2 \) for \( T_{5,7} \)).

Lemma 7: \( qn(T_{2h+5,k}) = 2 \) for \( h \geq 0 \) and \( k \geq 2h + 5 \).

Proof: Obviously, \( T_{2h+5,k} \) contains \((2h + 5)k\) vertices and \(2(2h + 5)k\) edges. By Corollary 2, we partition \( T_{2h+5,k} \) into two edge-disjoint spanning subgraphs \( G_1 \) and \( G_2 \), such that their leveled-planar embeddings have a common induced order. For \( G_1 \), its leveled-planar embedding consists of \( 2k + h \) levels. In each level, the indicated line prepares \( 2h + 5 \) positions for arranging vertices and all positions are ranked in a top-down fashion. We now arrange vertices in two parallelgrams according to the following rules:

1. For the first parallelgram, the vertex \((i, j)\) is located in the \((h + i + 3)\)-th position of line \( \ell_{i+j+1} \) for \( i = 0, 1, \ldots, h + 2 \) and \( j = 0, 1, \ldots, k - 1 \).

2. For the second parallelgram, the vertex \((i, j)\) is located in the \((i - h - 2)\)-th position of line \( \ell_{i+2h+1-i+j} \) for \( i = h + 3, h + 4, \ldots, 2h + 4 \) and \( j = 0, 1, \ldots, k - 1 \).

Thus, we have the following leveled edges: \((i, j), (i, j + 1)\) for \( i = 0, 1, \ldots, 2h + 4 \) and \( j = 0, 1, \ldots, k - 1 \), and \((i, 0), (i, k - 1)\) for \( i = 0, 1, \ldots, h \) are wrapped edges, and \((i, 0) \leq (i, j) \leq (i, k - 1) \) for \( i = 0, 1, \ldots, h \). It is easy to verify that \( G_1 \) contains \((2h + 5)(k - 1) + (2h + 3)k = 2(2h + 5)k -(2k + 2h + 5)\) mesh edges and all these edges do not cross each other (see Fig. 5(a) for such an embedding of \( G_1 \) for \( T_{5,7} \)).

From the above construction, we know that \( G_2 \) contains \( 2k + 2h + 5 \) edges, where \((i, 0), (i, k - 1)\) and \((0, j), (2h + 4, j)\) for \( i = 0, 1, \ldots, 2h + 4 \) and \( j = 0, 1, \ldots, k - 1 \) are wrapped edges, and \((h + 2, j), (h + 3, j)\) for \( j = 0, 1, \ldots, k - 1 \) are splitting edges. Let \( \sigma \) be the induced order of \( G_1 \) under
arranged in line $\ell_2$, and all remaining vertices are consecutively arranged in line $\ell_3$. Note that $(1, k - 2)$ is the second rightmost vertex in the second topmost row of the first parallelogram, and $(h + 3, k - h - 3)$ is the $(h + 3)$-th rightmost vertex in the top row of the second parallelogram. Hence, there are $2k$ leveled edges between lines $\ell_1$ and $\ell_2$ of three types, as follows:

(i) wrapped edges $((0, j), (2h + 4, j))$ for $j = 0, 1, \ldots, k - 2$,
(ii) wrapped edges $((i, 0), (i, k - 1))$ for $i = 0, 1, \ldots, h + 2$, and
(iii) splitting edges $((h + 2, j), (h + 3, j))$ for $j = 0, 1, \ldots, k - h - 3$.

We first observe that for two wrapped edges $((0, j), (2h + 4, j))$ and $((0, j'), (2h + 4, j'))$ where $0 \leq j < j' \leq k - 2$, we have $(0, j) <_{\sigma} (0, j')$ and $(2h + 4, j) <_{\sigma} (2h + 4, j')$. This shows that all type-(i) edges do not cross each other. Moreover, for two type-(ii) edges $((i, 0), (i, k - 1))$ and $((i + 1, 0), (i + 1, k - 1))$ where $i = 1, 2, \ldots, h + 1$, there is a type-(i) edge $((0, i), (2h + 4, i))$ such that $(i, 0) <_{\sigma} (i + 1, 0)$ (particularly, $(i, 0) = (0, i)$ if $i = 0$) and $(i, k - 1) <_{\sigma} (2h + 4, i) <_{\sigma} (i + 1, k - 1)$. Similarly, for two splitting edges $((h + 2, j), (h + 3, j))$ and $((h + 2, j + 1), (h + 3, j + 1))$ where $j = 1, 2, \ldots, k - h - 3$, there is a type-(i) edge $((0, h + j + 2), (2h + 4, h + j + 2))$ such that $(h + 2, j) <_{\sigma} (0, h + j + 2) <_{\sigma} (h + 2, j + 1)$ and $(h + 3, j) <_{\sigma} (2h + 4, h + j + 2) <_{\sigma} (h + 3, j + 1)$. That is, if we consider edges between $\ell_1$ and $\ell_2$ in a bottom-up fashion, then type-(i) edges and type-(ii) edges (respectively, type-(iii) edges) appear alternately, and all type-(ii) edges turn up before type-(iii) edges except $((h + 2, 0), (h + 3, 0))$ before $((h + 2, 0), (h + 2, k - 1))$. However, the exception does not produce crossing edges. Therefore, all edges between $\ell_1$ and $\ell_2$ do not cross each other.

We now consider the remaining $2h + 5$ edges between lines $\ell_2$ and $\ell_3$ of three types, as follows:

(i') wrapped edge $((0, k - 1), (2h + 4, k - 1))$;
(ii') wrapped edges $((i, 0), (i, k - 1))$ for $i = h + 3, h + 4, \ldots, 2h + 4$;
(iii') splitting edges $((h + 1, j), (h + 2, j))$ for $j = k - h - 1, \ldots, k - 1$.

It is clear that for two wrapped edges $((i, 0), (i, k - 1))$ and $((i', 0), (i', k - 1))$ where $h + 3 \leq i < i' \leq 2h + 4$, we have $<_{\sigma} (i', 0)$ and $<_{\sigma} (i, k - 1)$. Thus, all type-(ii') edges do not cross each other. Moreover, if we consider edges between $\ell_2$ and $\ell_3$ in a bottom-up fashion, it is easy to see that $((0, k - 1), (2h + 4, k - 1))$ is the first edge, and type-(ii') edges and type-(iii') edges appear alternately. Consequently, all edges between $\ell_2$ and $\ell_3$ do not cross each other (see Fig. 5(b) for such an embedding of $G_2$ for $T_{5,5}$).

We remark that a web site containing more instances of arched leveled-planar embeddings of 2-dimensional toroidal grids is available at URL http://poterp.iem.mcut.edu.tw/torus. From Lemmas 5, 6, and 7 and the symmetry of dimensions in a toroidal grid, we have the following result.
4. Upper Bounds on $qn(T_{k_1,k_2,...,k_n})$

In this section, we provide upper bounds on queue number for high dimensional toroidal grids. Given a vertex ordering $\sigma$ of a graph $G$, the length of an edge $(u,v) \in E(G)$ is defined to be $|\sigma(u) - \sigma(v)|$. Note that if $|\ell_{x}(x,y) - \ell_{y}(x,y)| \leq 1$, then $(u,v)$ and $(x,y)$ do not nest. Let $\sigma_1$ and $\sigma_2$ be two vertex orderings with no common vertex. The concatenation of $\sigma_1$ and $\sigma_2$, denoted $(\sigma_1 \circ \sigma_2)$, is the ordering $\sigma_1$ followed by the ordering $\sigma_2$.

For the $n$-dimensional toroidal grid $T_{k_1,k_2,...,k_n}$, we denote $(T_{k_1,k_2,...,k_n})^i$, $i = 0, 1, \ldots, k_n - 1$, as the subgraph of $T_{k_1,k_2,...,k_n}$ induced by the set of vertices with $i$ as the last digit in their labels. For notational convenience, if a vertex $v$ is contained in $T_{k_1,k_2,...,k_n}$, we simply write $v \in T_{k_1,k_2,...,k_n}$ instead of $v \in V(T_{k_1,k_2,...,k_n})$. Clearly, each subgraph $(T_{k_1,k_2,...,k_n})^i$ is isomorphic to $T_{k_1,k_2,...,k_n}$ under the isomorphism $\varphi((x_1,x_2,...,x_{n-1},i)) = (x_1,x_2,...,x_{n-1})$. The following lemma provides an induction to derive upper bounds.

**Lemma 9:** $qn(T_{k_1,k_2,...,k_n}) \leq qn(T_{k_1,k_2,...,k_{n-1}}) + 2$ if $n \geq 2$ and $k_i \geq 3$ for $i = 1, 2, \ldots, n$.

**Proof:** Let $\sigma$ be a vertex ordering of $T_{k_1,k_2,...,k_n}$ under which $T_{k_1,k_2,...,k_n}$ can be laid out using exactly $qn(T_{k_1,k_2,...,k_n})$ queues. From the isomorphism of $(T_{k_1,k_2,...,k_n})^1$ and $(T_{k_1,k_2,...,k_n})^0$, let $\sigma_1$ be the vertex ordering of $(T_{k_1,k_2,...,k_n})^1$ corresponding to $\sigma$ in $T_{k_1,k_2,...,k_n}$. Also, define $\Pi = (\sigma_0) \circ (\sigma_1) \circ \ldots \circ (\sigma_{k_n-1})$ to be the vertex ordering of $T_{k_1,k_2,...,k_n}$.

We now show that if vertices of $T_{k_1,k_2,...,k_n}$ are arranged as $\Pi$, then all edges can be partitioned into $qn(T_{k_1,k_2,...,k_n}) + 2$ queues without nested edges. Obviously, edges contained within a subgraph $(T_{k_1,k_2,...,k_n})^i$ for all $i = 0, 1, \ldots, k_n - 1$ can be laid out using $qn(T_{k_1,k_2,...,k_n})$ queues. Moreover, the remaining edges have two different lengths, one is the set of edges $\{(u,v) : u \in (T_{k_1,k_2,...,k_n})^i \text{ and } v \in (T_{k_1,k_2,...,k_n})^{i+1} \text{ for } i = 0, 1, \ldots, k_n - 2\} $ and the other is the set $\{(u,v) : u \in (T_{k_1,k_2,...,k_n})^0 \text{ and } v \in (T_{k_1,k_2,...,k_n})^{k_n-1}\}$. Therefore, they can be placed in two additional queues. □

Applying Theorem 8 and Lemma 9, we immediately obtain the following theorem.

**Theorem 10:** $qn(T_{k_1,k_2,...,k_n}) \leq 2n - 2$ if $n \geq 2$ and $k_i \geq 3$ for $i = 1, 2, \ldots, n$.

Since the family of toroidal grids contains $k$-ary $n$-cubes as a subfamily, we have the following corollary.

**Corollary 11:** $qn(Q_k^n) \leq 2n - 2$ for $n \geq 2$ and $k \geq 3$.

Note that the result of Corollary 11 improves Wood’s upper bound of generalized toroidal grids when $m = 1$ in Theorem 4. Also, if $n \geq 2$ and $k \geq 9$, the result is also an improvement over a recent result of [26] shown that $qn(Q_k^n) \leq 2n - 1$ for $n \geq 1$ and $k \geq 9$. In addition, since it has been shown in [26] that $qn(Q_k^n) \leq 2n - 3$ if $n \geq 3$ by constructing a 3-queue layout of $Q_k^n$, a natural way to improving the upper bound on queue number of toroidal grids is to settle the following conjecture.

**Conjecture 1:** $qn(T_{k_1,k_2,k_3}) \leq 3$ if $k_1, k_2, k_3 \geq 3$.

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