On 2-Rainbow Domination in Generalized Petersen Graphs

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Abstract

In this paper, we show that $\gamma_r^2(\text{GP}(n, k)) \leq n - 1$ for all $n \geq 13$ except $n = 2k + 2$, where $\text{GP}(n, k)$ is a generalized Petersen graph and $\gamma_r^2(\text{GP}(n, k))$ is the 2-rainbow domination number of $\text{GP}(n, k)$. We also conjecture that $\gamma_r^2(\text{GP}(n, k)) = n$ if and only if $n = 2k + 2$ for $k \geq 3$ or $(n, k) \in \{(5, 2), (7, 2), (7, 3), (10, 3), (11, 3), (11, 4)\}$.

Keywords: Domination; Rainbow domination; Generalized Petersen graphs.

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1. Introduction

Let $G = (V, E)$ be a graph, where $V(G)$ and $E(G)$ (simply written $V$ and $E$, respectively) are the vertex and edge, respectively, sets of $G$. Denote by $N(v) = \{u \in V \mid uv \in E\}$ the open neighborhood of $v$. The rainbow domination problem was introduced by Brešar, Henning and Rall in [3].

Definition 1.1. Let $f$ be a function which assigns to each vertex a set of colors chosen from the color set $\mathcal{C} = \{1, \ldots, k\}$. If $\bigcup_{u \in N(v)} f(u) = \mathcal{C}$ for each vertex $v \in V$ with $f(v) = \emptyset$, then $f$ is a $k$-rainbow dominating function ($k$-RDF for short) of $G$. The weight of a function $f$, denoted by $w(f)$, is equal to $\sum_{v \in V} |f(v)|$. Given a graph $G$, the minimum weight among all weights of $k$-RDFs is the $k$-rainbow domination number of $G$ and is denoted by $\gamma_{rk}(G)$.

The $k$-rainbow domination problem has been extensively studied [2–4, 6–8, 10–12]. Clearly, when $k = 1$ this concept coincides with the ordinary domination problem.

For two relatively prime natural numbers $n$ and $k$ with $n \geq 3$ and $1 \leq k \leq [(n - 1)/2]$, the generalized Petersen graph $GP(n, k)$ is a graph on $2n$ vertices with $V(GP(n, k)) = \{u_i, v_i \mid 1 \leq i \leq n\}$ and $E(GP(n, k)) = \{u_iu_{i+1}, u_iv_i, v_{i+k}v_{i+k+1} \mid 1 \leq i \leq n\}$ with subscripts modulo $n$ [1, 5, 9]. Hereafter, all operations on the subscripts of vertices are taken modulo $n$ unless stated otherwise. Moreover, if a subscript of a vertex is 0, then we will use $n$ instead.

In [2], Brešar and Šumenjak proved that $\gamma_{r2}(GP(n, k))$ is bounded above by $n$ and showed that $\gamma_{r2}(GP(5, 2)) = 5$. They also posed the following two open questions:

1. Is $\gamma_{r2}(GP(n,k)) = n$ for all $k \geq 2$ when $n = 2k + 1$?

2. Is $\gamma_{r2}(GP(n,3)) = n$ for all $n \geq 7$ when $n$ is not divisible by 3?

The first question was answered in negative by Tong et al. in [7]. The second question was answered also in negative by Xu in [12]. In [8], Wang and Wu gave a tight upper bound of $\gamma_{r2}(GP(n,k))$ when $n \geq 4k + 1$ and $k \geq 4$. All those aforementioned results show that $\gamma_{r2}(GP(n,k)) < n$ for $n \geq 4k + 1$ and $k \geq 2$. Thus it is interesting to find out the subclass of generalized Petersen graphs with $\gamma_{r2}(GP(n,k)) = n$. Some related results on $\gamma_{r2}(GP(n,k))$ are
summarized in Table 1.

### Table 1: Related results on $\gamma_{r2}(\text{GP}(n,k))$

<table>
<thead>
<tr>
<th>Reference No.</th>
<th>the ranges of $n$ and $k$</th>
<th>$\gamma_{r2}(\text{GP}(n,k))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$n = 5, k = 2$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) = n$</td>
</tr>
<tr>
<td>2</td>
<td>$n \geq 7, k \geq 2$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) \leq n$</td>
</tr>
<tr>
<td>7</td>
<td>$n = 7, k = 2$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) = n$</td>
</tr>
<tr>
<td>7</td>
<td>$n \geq 9, k = 2$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) &lt; n$</td>
</tr>
<tr>
<td>12</td>
<td>$n \geq 13, k = 3$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) &lt; n$</td>
</tr>
<tr>
<td>8</td>
<td>$n \geq 4k + 1, k \geq 4$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) &lt; n$</td>
</tr>
<tr>
<td>this paper</td>
<td>$\max{11, 2k + 2} &lt; n &lt; 4k + 1, k \geq 3$</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) &lt; n$</td>
</tr>
<tr>
<td>our conjecture</td>
<td>$n = 2k + 2, k \geq 3$ or</td>
<td>$\gamma_{r2}(\text{GP}(n,k)) = n$</td>
</tr>
<tr>
<td></td>
<td>$(n, k) \in {(5, 2), (7, 2), (7, 3), (10, 3), (11, 3), (11, 4)}$</td>
<td></td>
</tr>
</tbody>
</table>

In Figure 1, every square represents a generalized Petersen graph. The squares with light-grey color indicate those $\text{GP}(n,k)$ investigated in this paper and the squares with white color indicate those $\text{GP}(n,k)$ in our conjecture.

![Figure 1: The class of $\text{GP}(n,k)$ investigated in this paper.](image)

The rest of this paper is organized as follows. In Section 2, some preliminaries of $\gamma_{r2}(\text{GP}(n,k))$ are introduced. In Section 3, we show that $\gamma_{r2}(\text{GP}(n,k)) \leq n - 1$ for all even $n \geq 13$ with $n \neq 2k + 2$. In Section 4, we show that $\gamma_{r2}(\text{GP}(n,k)) \leq n - 1$ for all odd $n \geq 13$. Finally, Section 5 contains our concluding remarks.
2. Preliminaries

We follow the terms defined in [8]. For ease of understanding, we introduce some of them as follows. Let $f$ be a 2-RDF of $GP(n,k)$. The colors assigned to vertex pairs $u_\ell$ and $v_\ell$ for $i \leq \ell \leq j$ are denoted by $C_{i,j}$ which can be represented as a $2 \times (j-i+1)$ array as follows:

$$C_{i,j} = \left( \begin{array}{cccc} f(u_i) & f(u_{i+1}) & \cdots & f(u_j) \\ f(v_i) & f(v_{i+1}) & \cdots & f(v_j) \end{array} \right).$$

We call $f(u_i)$ and $f(v_i)$ a colored pair. For brevity, we also say that vertices $u_\ell$ and $v_\ell$ are in $C_{i,j}$ if $i \leq \ell \leq j$. The complement of a colored pair $(\alpha)^b_i$, denoted by $(\overline{\alpha})^b_i$, is defined as follows:

$$(\overline{\alpha})^b_i = \begin{cases} (1) & \text{if } (\alpha)^b_i = (2) \\ (2) & \text{if } (\alpha)^b_i = (1) \\ (2) & \text{if } (\alpha)^b_i = (1) \\ (1) & \text{if } (\alpha)^b_i = (2). \end{cases}$$

Let $X = C_{i,j}$. The complement of $X$, denoted by $\overline{X}$, is defined as follows:

$$\overline{X} = \left( \begin{array}{cccc} \overline{f(u_i)} & \overline{f(u_{i+1})} & \cdots & \overline{f(u_j)} \\ \overline{f(v_i)} & \overline{f(v_{i+1})} & \cdots & \overline{f(v_j)} \end{array} \right).$$

The concatenation of $X$ and $Y$ is denoted by $X \circ Y$, and the repetition of $X$ $m$ times is denoted by $X^m$.

Propositions 2.1 and 2.2 describe some basic coloring to satisfy $\bigcup_{y \in N(x)} f(y) = \{1,2\}$ for some vertex $x$ in $GP(n,k)$.

**Proposition 2.1.** Let $f$ be a coloring function of $GP(n,k)$ with $C_{i,i} = \binom{1}{0}$ or $\binom{2}{0}$. If either $C_{i-k,i-k} = \overline{C}_{k,i}$ or $C_{i+k,i+k} = \overline{C}_{i,i}$, then $\bigcup_{x \in N(u_i)} f(x) = \{1,2\}$.

**Proposition 2.2.** Let $f$ be a coloring function of $GP(n,k)$. If $C_{i,i+1}$ is in $\{(\binom{0}{10},\binom{0}{20})\}$ (respectively, $\{(\binom{2}{01},\binom{10}{2})\}$), then $\bigcup_{x \in N(u_i)} f(x) = \{1,2\}$ (respectively, $\bigcup_{x \in N(u_{i+1})} f(x) = \{1,2\}$).

The patterns depicted in Figures 2(a)-2(c) are called Patterns A, B, and C, respectively. Note that, in Patterns A, B, and C, we have that $\alpha = k + p + 2$ and $n = 2k + 2p + 3$ (respectively, $n = 2k + 2p + 4$) when $n$ is odd (respectively, even). Note also that since we only consider $n < 4k + 1$, it follows that $p \leq k - 2$. In Pattern A, we have that $C_{\alpha-2,\alpha+2} = \binom{0210}{10002}$ and $C_{\alpha-k-1,\alpha-k+2} = C_{\alpha+k-2,\alpha+k+1} = \binom{0001}{0120}$; in Pattern B, we have that $C_{\alpha-2,\alpha+2} = \binom{0210}{10002}$ and $C_{\alpha-k-1,\alpha-k+2} = C_{\alpha+k-2,\alpha+k+1} = \binom{0001}{0120}$; in Pattern C, we have that $C_{\alpha-2,\alpha+2} = \binom{0210}{10002}$ and $C_{\alpha-k-1,\alpha-k+2} = C_{\alpha+k-2,\alpha+k+1} = \binom{0001}{0120}$. 


Proposition 2.3 describes that some vertices $x$ in those patterns satisfy the condition that 
$$\bigcup_{y \in N(x)} f(y) = \{1, 2\}$$
by using only the colors assigned to the vertices in the patterns.

**Proposition 2.3.** In Patterns $A$ and $B$, we have that 
$$\bigcup_{y \in N(x)} f(y) = \{1, 2\}$$
for all $x \in V_1$ except $x \in \{v_{a-k-1}, v_{a+k+1}\}$. In Pattern $C$, we have that 
$$\bigcup_{y \in N(x)} f(y) = \{1, 2\}$$
for $x \in V_2$ except $x \in \{v_{a-k+1}, v_{a+k-1}\}$.

Our main idea for finding an upper bound of $\gamma_2(\text{GP}(n, k))$ is based on Patterns $A$, $B$, and $C$. We use $\ell_0(A)$ and $r_0(A)$ to denote vertices $v_{a-k-1}$ and $v_{a+k+1}$, respectively, in Pattern $A$; $\ell_0(B)$ and $r_0(B)$ denote vertices $v_{a-k-1}$ and $v_{a+k+1}$, respectively, in Pattern $B$; and $\ell_0(C)$ and $r_0(C)$ denote vertices $v_{a-k+1}$ and $v_{a+k-1}$ in Pattern $C$. Furthermore, let 
$$\ell_i \equiv \ell_{i-1} - k \pmod{n}$$
and 
$$r_i \equiv r_{i-1} + k \pmod{n}$$
for $i \in \{1, 2\}$ if exist. When the context is clear, $\ell_0(x)$ and $r_0(x)$ for $x \in \{A, B, C\}$ are simply written as $\ell_0$ and $r_0$, respectively. We
use Pos(ℓc) and Pos(rc) to denote the positions of ℓc and rc, respectively, for i ∈ {0, 1, 2} in GP(n, k). The possible positions of Pos(ℓc1), are in these ranges [k + p + 4, 2k + p − 1], [2k + p, 2k + p + 3], and [2k + p + 4, n] which are denoted by R1, R2, and R3, respectively (see Figure 3(a)). Note that the colored pair of Pos(ℓc2) (respectively, Pos(rc2)) is the same as that of Pos(ℓc0) (respectively, Pos(rc0)) while the colored pair of Pos(ℓc1) (respectively, Pos(rc1)) is the complement of that of Pos(ℓc0) (respectively, Pos(rc0)). After Pos(ℓc) and Pos(rc) for i ∈ {0, 1, 2} are determined, the remaining colored pairs are partitioned into blocks P_l, X_l, Y_l, Z_l, X_r, Y_r, Z_r, and P_r (see Figure 3(b)). Our task is to find out the colored pairs for those blocks such that γr2(GP(n, k)) < n.

Lemma 2.4 and Theorems 2.5-2.8 are some results in [2, 7, 8, 12].

**Lemma 2.4 ([2]).** For GP(n, k) with n ≥ 5, we have γr2(GP(n, k)) ≤ n.

**Theorem 2.5 ([7]).** For GP(n, 2) with n ≥ 5, we have

\[
\gamma_{r2}(GP(n, 2)) = \begin{cases} \\
\left\lceil \frac{4}{5}n \right\rceil & \text{if } n \equiv 3, 9 \pmod{10} \\
\left\lceil \frac{4}{5}n \right\rceil + 1 & \text{otherwise.}
\end{cases}
\]
Theorem 2.6 ([12]). For $GP(n, 3)$ with $n \geq 13$, we have

$$
\gamma_{r2}(GP(n, 3)) \leq \left\lceil \frac{7n}{8} \right\rceil + 1 \quad \text{if } n \equiv 1, 3, 8, 9, 10, 11, 12 \pmod{16}.
$$

Theorem 2.7 ([8]). For $GP(n, k)$ with $k$ even and $n \geq 4k + 1$, we have

$$
\gamma_{r2}(GP(n, k)) \leq \left\lceil \frac{2kn}{2k+1} \right\rceil + 1 \quad \text{if } n \equiv k + 1, k + 3, \ldots, 2k - 1, 4k + 1 \pmod{4k+2}.
$$

Theorem 2.8 ([8]). For $GP(n, k)$ with $n \geq 4k + 1$ and odd $k \neq 3$, we have

$$
\gamma_{r2}(GP(n, k)) \leq \left\lceil \frac{2kn}{2k+1} \right\rceil + 1 \quad \text{if } n \equiv 0, \ldots, k - 4, k - 2, k + 1, k + 2, \ldots, 2k - 3, 2k - 1, 2k, 4k + 1 \pmod{4k+2},
$$

where

$$
S = \{0, 3, 4, \ldots, k - 4, k - 2, k + 1, k + 2, \ldots, 2k - 3, 2k - 1, 2k, 4k + 1\}.
$$

By Theorem 2.5, the exact values of $\gamma_{r2}(GP(n, 2))$ for $n \geq 5$ are known. By Theorems 2.6-2.8, we know that $\gamma_{r2}(GP(n, k)) \leq n - 1$ when $n \geq 4k + 1$. In this paper, we investigate upper bounds of $\gamma_{r2}(GP(n, k))$ for $n \geq 13$ and $n \neq 2k + 2$. In particular, we focus on $13 \leq n < 4k + 1$ in the rest of this paper.

3. An upper bound of $\gamma_{r2}(GP(n, k))$ for even $n \geq 14$

In this section, we consider the case where $n$ is even. Moreover, since $n$ and $k$ are relatively prime, it follows that $k$ must be odd. Recall that we only consider the case where $\max\{11, 2k + 2\} < n < 4k + 1$ for $k \geq 3$ in the rest of this paper. Note that, when $n = 12$ and $k \geq 3$, only $GP(12, 5)$ is a feasible case. However, $GP(12, 5)$ satisfies $n = 2k + 2$. Thus the smallest value of $n$ is actually 14 and $k \geq 5$ when $n$ is even. Thus, in this section, we investigate $\gamma_{r2}(GP(n, k))$ for even $n \geq 14$, $k \geq 5$, and $n \neq 2k + 2$. Since $n$ is even and $n < 4k + 1$, we may assume that $n = 2(k + p) + 4$ with $0 \leq p \leq k - 3$. Note that, when $p = k - 2$, we have $n = 4k$ which is not relatively prime to $k$. Thus the largest value of $p$ is $k - 3$.

In Lemmas 3.1-3.3, we show that $\gamma_{r2}(GP(n, k)) \leq n - 1$ when $\ell c_1(A) \in R_3$, $\ell c_1(A) \in R_2$, and $\ell c_1(A) \in R_1$, respectively.

**Lemma 3.1.** If $\ell c_1(A) \in R_3$, then $\gamma_{r2}(GP(n, k)) \leq n - 1$, where $n = 2(k + p) + 4 \geq 14$ with odd $k \geq 5$ and $0 \leq p \leq k - 3$. 

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Proof. Based on Pattern A, we only set the colored pair in $\text{Pos}(\ell c_1)$ to $\binom{p}{2}$. After that, we have blocks $X_l = C_{p+5,k+p-1}$, $X_r = C_{k+p+5,2k+p-1}$, $Y_r = C_{2k+p+4,k+3p+4}$, and $Z_r = C_{k+3p+6,n}$ (see Figure 4). The coloring function $f$ for those blocks are set as follows: $P_l = \binom{10}{0}n^2/2$ if $p$ is even and $P_l = \binom{10}{2}r(10)(p-1)/2$ if $p$ is odd, $X_l = \binom{10}{0}(k-5)/2$, $X_r = \mathcal{X}_l$, $Y_r = \mathcal{C}_{k+p+4,k+3p+4+|Y_r|}$, and $Z_r = \mathcal{C}_{p+2,p+1+|Z_r|}$. This completes the description of the coloring function $f$. To show that $\gamma_2(\text{GP}(n,k)) \leq n-1$, we consider the following two cases.

Case 1. $X_l$ is empty.

If $X_l$ is empty, then $k+p = p+5$, namely $k = 5$. By the assumption $p \leq k - 3$, this yields $p \leq 2$. Since $\ell c_1(A) \in R_3$, we have $k \leq 2p + 1$ which yields $p \geq 2$. As a consequence, we have $p = 2$. Thus both $X_r$ and $Y_r$ are also empty. Thus, in this case, we only need to consider $\text{GP}(18,5)$. By the coloring function $f$, we have $P_l = \binom{10}{0}$ and $P_r = \binom{10}{2}/(10)$, namely $C_{1,18} = \binom{1002010201020010201020010201020102010201}{020102001020102010201020102010201}$. It is easy to verify that the coloring function $f$ for $\text{GP}(18,5)$ is a 2-RDF and $\text{GP}(18,5) \leq 17$. Thus this case holds.

Case 2. $X_l$ is not empty.

First, we have to show that $C_{k+p-1} = \binom{10}{p}$ which is the colored pair in position $\text{Pos}(rc_1)$. If $p$ is even (respectively, odd), then $k+p-1$ is even (respectively, odd). By the colored pairs assigned to $P_l$, we can find that $C_{k+p-1} = \binom{10}{p}$. Since $|Y_r| = k + 3p + 4 - (2k + p + 4) + 1 = -k + 2p + 1$ is an even number, by the colored pairs assigned to $Y_r$, we have that $f(u_{k+p+4}) = \{1\}$. Thus we have to make sure that $f(u_{k+3p+6}) = \{2\}$ so that $\bigcup_{y \in N(u_{k+3p+5})} f(y) = \{1,2\}$. By the colored pairs assigned to $Z_r$, we can find that $C_{k+3p+6,k+3p+6} = \mathcal{C}_{p+2} = \binom{10}{p}/(p)$. This further implies that $\bigcup_{y \in N(u_{k+3p+5})} f(y) = \{1,2\}$. By Propositions 2.1, 2.2, and 2.3, we can find that $\bigcup_{y \in N(x)} f(y) = \{1,2\}$ for all the other vertices $x \in V(\text{GP}(n,k))$ with $f(x) = \emptyset$. This implies that $f$ is a 2-RDF of $\text{GP}(n,k)$ with $w(f) = n-1$ and the lemma follows. \qed

Figure 4: An illustration for Lemma 3.1.
Lemma 3.2. If \( \ell c_1(A) \in R_2 \), then \( \gamma_{r2}(\text{GP}(n,k)) \leq n - 1 \), where \( n = 2(k + p) + 4 \geq 14 \) with odd \( k \geq 5 \) and \( 0 \leq p \leq k - 3 \).

**Proof.** Since \( k \) is odd and \( \ell c_1(A) \in R_2 \), we only need to consider the case where \( k \in \{2p + 3, 2p + 5\} \). For the case where \( k = 2p + 3 \) (respectively, \( k = 2p + 5 \)), it follows that \( n = 2(k + p) + 4 = 3k + 1 \) (respectively, \( n = 3k - 1 \)). Note that \( \text{GP}(n,k) \) is isomorphic to \( \text{GP}(n,3) \) when \( n \in \{3k - 1, 3k + 1\} \). By Theorem 2.6, we have that \( \gamma_{r2}(\text{GP}(n,k)) = \gamma_{r2}(\text{GP}(n,3)) \leq n - 1 \) when \( n \geq 13 \). Thus this lemma holds. \( \square \)

Lemma 3.3. If \( \ell c_1(A) \in R_1 \), then \( \gamma_{r2}(\text{GP}(n,k)) \leq n - 1 \), where \( n = 2(k + p) + 4 \geq 14 \) with odd \( k \geq 5 \) and \( 0 \leq p \leq k - 3 \).

**Proof.** Based on Pattern \( A \), it is easy to check that \( \text{Pos}(\ell c_0) = p + 1, \text{Pos}(\ell c_1) = p + 1 - k \equiv k + 3p + 5 \pmod{n}, \text{Pos}(\ell c_2) = 3p + 5, \text{Pos}(rc_0) = 2k + p + 3, \text{Pos}(rc_1) = 3k + p + 3 \equiv k - p - 1 \pmod{n}, \) and \( \text{Pos}(rc_2) = 2k - p - 1 \) (see Figure 5). First we show the existence of \( X_l, Y_l, Z_l, X_r, Y_r, \) and \( Z_r \) even though they might be empty. We claim that

1. \( \text{Pos}(\ell c_2) \neq \text{Pos}(rc_1) \),
2. \( \text{Pos}(rc_1) > p + 4 \), and
3. \( \text{Pos}(\ell c_2) > p + 4 \).

For the first claim, if \( \text{Pos}(\ell c_2) = 3p + 5 = k - p - 1 = \text{Pos}(rc_1) \), then this yields \( k = 4p + 6 \) which contradicts that \( k \) is odd. Thus the first claim holds. For the second claim, if \( \text{Pos}(rc_1) = k - p - 1 \leq p + 4 \), then this yields \( k \leq 2p + 5 \) which contradicts the assumption that \( \ell c_1(A) \in R_1 \), namely \( k > 2p + 5 \). Thus the second claim holds. Since \( \text{Pos}(\ell c_2) = 3p + 5 \), it is clear that the last claim holds.

Note that \( |X_l| = |Z_l| \), where \( |X_l| \) (respectively, \( |Z_l| \)) denotes the number of colored pairs in \( X_l \) (respectively, \( Z_l \)). This implies that all of \( \text{Pos}(\ell c_i) \) and \( \text{Pos}(rc_i) \) for \( i \in \{0, 1, 2\} \) are different. This further implies the existence of \( X_l, Y_l, Z_l, X_r, Y_r, \) and \( Z_r \). According to the values of \( \text{Pos}(\ell c_2) \) and \( \text{Pos}(rc_1) \), we distinguish the following two cases.
Case 1. \( \text{Pos}(\ell c_2) > \text{Pos}(rc_1) \)

In this case, we have that \( X_l = C_{p+5,k-p-2}, Y_l = C_{k-p,3p+4}, Z_l = C_{3p+6,k+p-1}, X_r = C_{k+p+5,2k-p-2}, Y_r = C_{2k-p,k+3p+4}, \) and \( Z_r = C_{k+3p+6,2k+p-1} \) (see Figure 5(a)). It is easy to verify that only \( Y_l \) or \( Y_r \) might be empty. The function \( f \) of \( \text{GP}(n,k) \) is defined as follows: Set \( X_l = C_{p+5,k-p-2} = (\ell)^0 \odot C_{k-p,3p+4} = (\ell)^0 \odot (4p-k+5)/2, Z_l = C_{3p+6,k+p-1} = (\ell)^0 \odot (20)_{(20)}^k/2, X_r = \overline{X}_l, Y_r = \overline{Y}_l, Z_r = \overline{Z}_l, P_l = C_{1,p} = \overline{C}_{k+1,k+p}, \) and \( P_r = C_{n-p,n} = \overline{C}_{k+p+4,k+2p+4} \) (see Figure 5(a)). Recall that \((01)^{(k-2p-7)/2}\) means that the pattern \((01)^{(k-2p-7)/2}\) repeats \((k-2p-7)/2\) times and \((01)^{(20)}_{(20)}^{k-2p-7)/2}\) denotes the concatenation of the patterns \((01)^{(20)}_{(20)}^{k-2p-7)/2}\) and \((01)^{(01)}_{(20)}^{k-2p-7)/2}\). This completes the description of \( f \).

It remains to show that \( f \) is a 2-RDF of \( \text{GP}(n,k) \). By Proposition 2.3 and the construction of \( f \), we know that \( \bigcup_{y \in \text{N}(x)} f(y) = \{1, 2\} \) for all \( x \in V_1 \). By Proposition 2.2, we have that \( \bigcup_{y \in \text{N}(u_i)} f(y) = \{1, 2\} \) for \( i \in \{1, \ldots, n\} \) when \( f(u_i) = \emptyset \). Since all of \( X_l, Y_l, \) and \( Z_l \) have their corresponding \( X_r = \overline{X}_l, Y_r = \overline{Y}_l, \) and \( Z_r = \overline{Z}_l \), by Proposition 2.1, it follows that \( \bigcup_{x \in \text{N}(v_i)} f(x) = \{1, 2\} \) for all \( v_i \in X_l \cup Y_l \cup Z_l \cup X_r \cup Y_r \cup Z_r \) with \( f(v_i) = \emptyset \). Since \( P_l = C_{k+1,k+p} \) (respectively, \( P_r = C_{k+p+4,k+2p+4} \)), we have that \( \bigcup_{x \in \text{N}(v_i)} f(x) = \{1, 2\} \) for all \( v_i \in P_l \) (respectively, \( v_i \in P_r \)) with \( f(v_i) = \emptyset \).

Now we consider the case where \( u_i \in P_l \cup P_r \). We can find that \( C_{p+4,p+5} = (11)^0 \) (respectively, \( C_{2k+p-1,2k+p} = (20)^0 \) ). This results in \( C_{1,2} = (20)^0_{(20)} \) and \( C_{n,n} = (2)^0 \) (respectively, \( C_{n-1,n} = (00)^0_{(11)} \) and \( C_{1,1} = (01)^0_{(1)} \)) when \( p = k-3 \) (respectively, \( p = k-4 \)). Thus \( f \) is not a 2-RDF of \( \text{GP}(n,k) \) when \( p = k-3 \) or \( p = k-4 \). Note that \( \ell c_1(A) \in R_1 \) implies \( k > 2p + 5 \). If \( p = k-3 \) (respectively, \( p = k-4 \)), then, after replacing \( p = k-3 \) in the inequality \( k > 2p + 5 \), this yields \( k < 1 \) (respectively, \( k < 3 \)) which contradicts the assumption that \( k \geq 5 \). Thus these two cases are impossible. For all the other cases, it is easy to verify that \( \bigcup_{x \in \text{N}(u_i)} f(x) = \{1, 2\} \) for all \( u_i \in P_l \cup P_r \) with \( f(u_i) = \emptyset \). This implies that \( f \) is a 2-RDF of \( \text{GP}(n,k) \) with \( w(f) = n-1 \).

Case 2. \( \text{Pos}(\ell c_2) < \text{Pos}(rc_1) \).

In this case, we have that \( X_l = C_{p+5,3p+4}, Y_l = C_{3p+6,k-p-2}, Z_l = C_{k-p,k+p-1}, X_r = C_{k+p+5,k+3p+4}, Y_r = C_{k+3p+6,2k-p-2}, \) and \( Z_r = C_{2k-p,2k+p-1} \) (see Figure 5(b)). The function \( f \) of \( \text{GP}(n,k) \) is defined as follows: Set \( X_l = C_{p+5,3p+4} = (20)^0_{(20)} p, Y_l = C_{3p+6,k-p-2} = (02)^0_{(10)}(k-4p-3)/2, \)
\[ Z_t = C_{k-p,k+p-1} = \left( \frac{10}{11} \right)^p, \quad X_r = X_l, \quad Y_r = Y_l, \quad Z_r = Z_l, \quad P_l = C_{1,p} = C_{k+1,k+p}, \text{ and } P_r = C_{n-p,n} = C_{k+p+4,k+2p+4} \text{ (see Figure 5(b)). This completes the description of } f \text{ for this case.} \]

By using a similar argument as in Case 1, we can find that \( \bigcup_{y \in N(x)} f(y) = \{1, 2\} \) for all vertices \( x \) in \( GP(n,k) \) with \( f(x) = \emptyset \) except when \( p = 0 \) and \( k = 7 \). Note that \( n = 18 \) when \( p = 0 \) and \( k = 7 \). Moreover, \( GP(18,7) \) is isomorphic to \( GP(18,5) \). By Case 1 of Lemma 3.1, we can also derive a 2-RDF of \( GP(18,7) \) with \( w(f) = n - 1 \).

Finally, we claim that the pattern \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) cannot occur at \( C_{1,2} \). Suppose to the contrary that \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) occurs at \( C_{1,2} \). This case occurs only when (1) \( Y_l \) is empty and \( 1 + k = 3p + 5 \), or (2) \( Y_l \) is not empty and \( 1 + k = k - p - 2 \). The former results in \( k = 3p + 4 \). This yields \( Pos(rc_1) = k - p - 1 = 2p + 3 < 3p + 5 = Pos(\ell c_2) \) which contradicts the assumption \( Pos(\ell c_2) < Pos(rc_1) \). For the latter case, this yields \( p = -1 \) which contradicts the assumption that \( p \geq 0 \). Similarly, we can show that the pattern \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) cannot occur at \( C_{n-1,n} \). This completes the proof. \( \square \)

![Figure 5: Illustrations for Lemma 3.3.](image)

4. An upper bound of \( \gamma_{r2}(GP(n,k)) \) for odd \( n \geq 13 \)

In this section, we consider odd \( n \) with \( 13 \leq n < 4k + 1 \) and \( k \geq 4 \). Note that, when \( k = 4 \), since \( n < 4k + 1 \), the feasible pairs \((n,k)\) are in \{\((13,4), (15,4)\)\}. Since \( GP(13,4) \) is isomorphic
to \(GP(13, 3)\), by Theorem 2.6, we have \(\gamma_r(\text{GP}(13, 3)) \leq 12\). For the case where \((n, k) = (15, 4)\), the colored paired \(C_{1,15} = (0200202010100101010)\) is a 2-RDF for \(\text{GP}(15, 4)\) with \(\gamma_r(\text{GP}(15, 4)) \leq 14\). Thus, in the rest of this section, we assume that \(n = 2(k + p) + 3\) with \(k \geq 5\) and \(0 \leq p < k - 1\). Based on the possible values of \(p\) and \(k\), we consider the following five cases on deriving upper bounds of \(\gamma_r(\text{GP}(n, k))\):

(i) \(p = 0\),

(ii) \(p \neq 0\) and \(5 \leq k < 2p\),

(iii) \(p \neq 0\) and \(2p \leq k \leq 2p + 4\),

(iv) \(p \neq 0\) and odd \(k > 2p + 4\), and

(v) \(p \neq 0\) and even \(k > 2p + 4\).

In Lemmas 4.1-4.5, we investigate upper bounds of \(\gamma_r(\text{GP}(n, k))\) for these cases.

**Lemma 4.1.** For \(\text{GP}(n, k)\) with \(n = 2k + 3 \geq 13\), we have \(\gamma_r(\text{GP}(n, k)) \leq n - 1\).

**Proof.** Based on Pattern \(B\), we define the coloring function \(f\) of \(\text{GP}(n, k)\) as follows. Set \(X_l = C_{5,k-1} = (01\binom{k-5}{20})\) if \(k\) is odd; otherwise, set \(X_l = C_{5,k-1} = (01\binom{k-6}{20})\) \(\circ (0\binom{2}{2})\) (see Figure 6). After that, set \(X_r = C_{k+5,2k-1} = X\). This completes the description of \(f\).

We need to prove that \(f\) is a 2-RDF of weight \(n - 1\). By Proposition 2.3, we know that \(\bigcup_{y \in N(x)} f(y) = \{1, 2\}\) for \(x \in V_1\). Since \(C_{k+5,2k-1} = C_{5,k-1}\), by Propositions 2.1 and 2.2, we have that \(\bigcup_{y \in N(x)} f(y) = \{1, 2\}\) for all \(x \in X_l \cup X_r\) when \(f(x) = \emptyset\). It remains to consider the case where \(f(v_j) = \emptyset\) for \(j \in \{1, 2k + 3\}\). It can be derived easily that \(\bigcup_{x \in N(v_1)} f(x) = \{f(u_1), f(v_{k+1}), f(v_{k+4})\} = \{1, 2\}\). Analogously, we can derive \(\bigcup_{x \in N(v_{2k+3})} f(x) = \{1, 2\}\). Therefore, the function \(f\) is a 2-RDF of \(\text{GP}(n, k)\) with \(n = 2k + 3\) and \(k \geq 5\). Since every colored pair has weight 1 except that \(w(f(u_{k+2})) + w(f(v_{k+2})) = 0\), this yields \(w(f) = n - 1\). This completes the proof. \(\square\)
Lemma 4.2. For $\text{GP}(n, k)$ with $n = 2(k+p)+3$ and $5 \leq k < 2p$, we have $\gamma_r(\text{GP}(n, k)) \leq n - 1$.

Proof. Based on Pattern $A$, we only set the colored pair in $\text{Pos}(rc_1)$ as in Lemma 3.1. After that, we have blocks $X_l = C_{p+5,k+p-1}$, $X_r = C_{k+p+5,2k+p-1}$, $Y_r = C_{2k+p+4,k+3p+3}$, and $Z_r = C_{k+3p+5,n}$. If $k$ is odd, then the coloring function $f$ for those blocks are set as follows: $P_l = \binom{10}{0_2}(p-1)/2 \cdot \binom{0_2}{0}$ if $p$ is odd and $P_l = \binom{0_2}{0_2}(p-2)/2 \cdot \binom{0}{0_2}$ if $p$ is even, $X_l = C_{p+5,k+p-1} = \binom{0_1}{0_2}(k-5)/2$, $X_r = X_l$, $Y_r = C_{k+p+4,3p+3}$, and $Z_r = C_{p+2,k}$ (see Figure 7(a)). If $k$ is even, then the coloring function $f$ for those blocks are set as follows: $P_l = \binom{0_2}{0_2}p/2$ if $p$ is even and $P_l = \binom{0_2}{0_2}(p-1)/2$ if $p$ is odd, $X_l = C_{p+5,k+p-1} = \binom{0_2}{0_2}(k-6)/2$, $X_r = X_l$, $Y_r = C_{k+p+4,3p+3}$, and $Z_r = C_{p+2,k}$ (see Figure 7(b)).

By using a similar argument as in Lemma 3.1, we can show that $\bigcup_{y \in N(x)} f(y) = \{1, 2\}$ for all vertices $x \in V(\text{GP}(n, k))$ with $f(x) = \emptyset$ and $w(f) = n - 1$. This completes the proof. \hfill $\square$

![Figure 6: An illustration for Lemma 4.1.](image)

![Figure 7: Illustrations for Lemma 4.2.](image)

Lemma 4.3. If $2p \leq k \leq 2p + 4$, then $\gamma_r(\text{GP}(n, k)) \leq n - 1$, where $n = 2(k+p)+3 \geq 13$ with $p > 0$ and $k \geq 5$. 
Proof. If \( k = 2p + 3 \), then \( n = 2(k + p) + 3 = 6p + 9 \). This implies that \( k \) is not relatively prime to \( n \) and there is no such \( \text{GP}(n,k) \). Thus we only need to consider \( k \in \{2p, 2p+1, 2p+2, 2p+4\} \).

For the case where \( k = 2p + 2 \) (respectively, \( k = 2p + 4 \)), it follows that \( n = 2(k + p) + 3 = 2k + 1 + 2p + 2 = 3k + 1 \) (respectively, \( n = 3k - 1 \)). This implies that those \( \text{GP}(n,k) \) with \( k \in \{2p+2, 2p+4\} \) are isomorphic to their corresponding \( \text{GP}(n,3) \). By Theorem 2.6, we have \( \gamma_{r2}(\text{GP}(n,k)) = \gamma_{r2}(\text{GP}(n,3)) \leq n - 1 \) when \( n \geq 13 \). Thus the lemma also holds for \( k \in \{2p+2, 2p+4\} \).

It remains to consider the case where \( k \in \{2p, 2p+1\} \). First, we consider the case where \( k = 2p + 1 \). If \( p = 1 \), then \( k = 3 \) which contradicts our assumption that \( k \geq 5 \). Thus we may assume that \( p \geq 2 \). Accordingly, based on Pattern \( C \), we define the coloring function \( f \) of \( \text{GP}(n,k) \) as follows: Set \( C_{p-1,p-1} = (\begin{pmatrix} 0 \end{pmatrix}) \) and \( C_{2k+p+5,2k+p+5} = (\begin{pmatrix} 0 \end{pmatrix}) \), where \( \text{Pos}(rc_1) = p - 1 \) and \( \text{Pos}(\ell c_1) = 2k+p+5 \). After that, set \( X_l = C_{p+4,k+p-1} = (\begin{pmatrix} 2 \end{pmatrix} \circ (\begin{pmatrix} 0 \end{pmatrix})^{(k-5)/2} \), \( X_r = C_{k+p+5,2k+p} = \overline{X}_l \), \( P_l = C_{1,p} = \overline{C}_{k+1,k+p} \), and \( P_r = C_{n-p+1,n} = \overline{C}_{k+p+4,k+2p+3} \) (see Figure 8(a)). Note that it is easy to check that \( |X_l| \) is odd. This completes the description of \( f \) for this case. Thus, by Propositions 2.1, 2.2, and 2.3, we can find that \( f \) is a \( 2\)-RDF of \( \text{GP}(n,k) \) with \( w(f) = n - 1 \).

Finally, we consider the case where \( k = 2p \geq 6 \). That is, \( n = 6p + 3 \). Based on Pattern \( A \), we define the coloring function \( f \) of \( \text{GP}(n,k) \) as follows. If \( p \) is even, then set \( P_l = C_{1,p} = (\begin{pmatrix} 1 \end{pmatrix})^{p/2} \) and \( P_r = C_{5p+4,n} = (\begin{pmatrix} 0 \end{pmatrix})^{p/2} \) (see Figure 8(b)); otherwise, set \( P_l = C_{1,p-1} = (\begin{pmatrix} 0 \end{pmatrix}) \circ (\begin{pmatrix} 0 \end{pmatrix})^{(p-1)/2} \) and \( P_r = C_{5p+4,n} = (\begin{pmatrix} 0 \end{pmatrix})^{(p-1)/2} \circ (\begin{pmatrix} 0 \end{pmatrix}) \). After that, set \( X_l = C_{p+5,2p} = \overline{C}_{5p+8,n}, Y_l = C_{2p+1,3p-1} = \overline{C}_{1,p-1}, X_r = C_{3p+5,4p+3} = \overline{C}_{5p+5,n}, \) and \( Y_r = C_{4p+4,5p-1} = \overline{C}_{1,p-4} \). Note that \( X_l \) and \( Y_r \) are empty when \( p \leq 4 \). Moreover, for the case where \( p = 3 \) and \( k = 6 \), it follows that \( n = 21 \) which is not relatively prime to \( k \). Thus \( k \geq 8 \) when \( p = 3 \). By Propositions 2.1, 2.2, and 2.3, it is easy to check that \( f \) is a \( 2\)-RDF of \( \text{GP}(n,k) \) with \( w(f) = n - 1 \). This completes the proof. \( \square \)
Lemma 4.4. If $k$ is odd and $k > 2p+4$, then $\gamma_{12}(\text{GP}(n,k)) \leq n-1$, where $n = 2(k+p)+3 \geq 13$ with $p > 0$ and $k \geq 5$.

Proof. Based on Pattern A, the reasoning of this lemma is almost the same as Lemma 3.3. Since $n$ is odd, we have $\text{Pos}(l_c_1) = k + 3p + 4$, $\text{Pos}(l_c_2) = 3p + 4$, $\text{Pos}(r_c_1) = k - p$, $\text{Pos}(r_c_2) = 2k - p$ (see Figure 9). We claim that $\text{Pos}(l_c_2) \neq \text{Pos}(r_c_1)$. For otherwise, it follows that $\text{Pos}(l_c_2) = 3p + 4 = k - p = \text{Pos}(r_c_1)$. This yields $k = 4p + 4$ which contradicts the assumption that $k$ is odd. Thus the claim holds. According to the values of $\text{Pos}(l_c_2)$ and $\text{Pos}(r_c_1)$, we distinguish the following two cases.

Case 1. $\text{Pos}(l_c_2) > \text{Pos}(r_c_1)$.

In this case, $X_l = C_{p+5,k-p-1}$, $Y_l = C_{k-p+1,3p+3}$, $Z_l = C_{3p+5,k+p-1}$, $X_r = C_{k+p+5,2k-p-1}$, $Y_r = C_{2k-p+1,k+3p+3}$, and $Z_r = C_{k+3p+5,2k+p-1}$ (see Figure 9(a)). Note that all of $X_l$, $Y_l$, and $Z_l$ might be empty. The function $f$ of $\text{GP}(n,k)$ is defined as follows: Set $X_l = \binom{01}{20(20)}(k-2p-5)/2$, $Y_l = \binom{10}{02}(4p-k+3)/2$, $Z_l = \binom{20}{01}(k-2p-5)/2$, $X_r = \overline{X_l}$, $Y_r = \overline{Y_l}$, $Z_r = \overline{Z_l}$, $P_l = C_{1,p} = C_{k+1,k+p}$, and $P_r = C_{n-p,n} = C_{k+p+4,k+2p+4}$. By using a similar argument as in Case 1 of Lemma 3.3, we can find that $f$ is a 2-RDF of $\text{GP}(n,k)$ with $w(f) = n-1$. Note that the numbers of colored pairs in $X_l$, $Y_l$, and $Z_l$ are all even in this case. Thus the patterns $\binom{00}{22}$ and $\binom{00}{11}$, which occur in Case 1 of Lemma 3.3, do not occur in this case.
Case 2. $\text{Pos}(\ell c_2) < \text{Pos}(rc_1)$.

In this case, we have $X_l = C_{p+5,3p+3}$, $Y_l = C_{3p+5,k-p-1}$, $Z_l = C_{k-p+1,k+p-1}$, $X_r = C_{k+p+5,k+3p+3}$, $Y_r = C_{k+3p+5,2k-p-1}$, and $Z_r = C_{2k-p+1,2k+p-1}$ (see Figure 9(b)). Set $X_l = (\binom{0}{2} \circ (\binom{20}{01})^{p-1}$, $Y_l = (\binom{02}{12})^{(k-4p-7)/2} \circ (\binom{91}{10})$, $Z_l = (\binom{10}{02})^{p-1} \circ (\binom{1}{0})$, $X_r = \overline{X_l}$, $Y_r = \overline{Y_l}$, $Z_r = \overline{Z_l}$, $P_l = C_{1,p} = C_{k+1,k+p}$, and $P_r = C_{n-p+1,n} = C_{k+p+4,k+2p+3}$. This completes the description of $f$ for this case.

By using a similar argument as in Case 2 of Lemma 3.3, we can find that $\bigcup_{y \in N(x)} f(y) = \{1, 2\}$ for all vertices $x$ in $\text{GP}(n, k)$. Thus $f$ is a 2-RDF of $\text{GP}(n, k)$ with $w(f) = n - 1$ after the adjustment. This completes the proof. 

\[\begin{array}{c|cc|cc|cc}
\hline
& \multicolumn{2}{c|}{X_l} & \multicolumn{2}{c|}{Y_l} & \multicolumn{2}{c}{Z_l} \\
\hline
P_l & 0 & 10 & 0 & 10 & 0 & 10 \\
0120 & 0120 \\
\hline
\end{array}\]

\[\begin{array}{c|cc|cc|cc}
\hline
& \multicolumn{2}{c|}{X_r} & \multicolumn{2}{c|}{Y_r} & \multicolumn{2}{c}{Z_r} \\
\hline
P_r & 0 & 10 & 0 & 10 & 0 & 10 \\
0120 & 0120 \\
\hline
\end{array}\]

(a) $\text{Pos}(\ell c_2) > \text{Pos}(rc_1)$

\[\begin{array}{c|cc|cc|cc}
\hline
& \multicolumn{2}{c|}{X_l} & \multicolumn{2}{c|}{Y_l} & \multicolumn{2}{c}{Z_l} \\
\hline
P_l & 0 & 10 & 0 & 10 & 0 & 10 \\
0120 & 0120 \\
\hline
\end{array}\]

\[\begin{array}{c|cc|cc|cc}
\hline
& \multicolumn{2}{c|}{X_r} & \multicolumn{2}{c|}{Y_r} & \multicolumn{2}{c}{Z_r} \\
\hline
P_r & 0 & 10 & 0 & 10 & 0 & 10 \\
0120 & 0120 \\
\hline
\end{array}\]

(b) $\text{Pos}(\ell c_2) < \text{Pos}(rc_1)$

Figure 9: Illustrations for Lemma 4.4.

**Lemma 4.5.** If $k$ is even and $k > 2p+4$, then $\gamma_{\ell 2}(\text{GP}(n, k)) \leq n - 1$, where $n = 2(k+p)+3 \geq 13$ with $p > 0$ and $k \geq 6$.

**Proof.** Based on Pattern A, the coloring function $f$ constructed in this lemma is almost the same as that in Lemma 4.4. However, $\text{Pos}(\ell c_2)$ might be equal to $\text{Pos}(rc_1)$. For the case where $\text{Pos}(\ell c_2) \geq \text{Pos}(rc_1)$. In Lemma 4.4 all of $X_l, Y_l, Z_l, X_r, Y_r,$ and $Z_r$ are of even number of colored pairs while, in this lemma, all of them are of odd number of colored pairs. We only need to add one more colored pair for each of them. That is, $X_l = (\binom{01}{20})^{(k-2p-6)/2} \circ (\binom{3}{2})$, $Y_l = (\binom{10}{02})^{(4p-k-2)/2} \circ (\binom{1}{0})$, $Z_l = (\binom{20}{01})^{(k-2p-6)/2} \circ (\binom{2}{0})$, $X_r = \overline{X_l}$, $Y_r = \overline{Y_l}$, $Z_r = \overline{Z_l}$, $P_l = C_{1,p} = C_{k+1,k+p}$, and $P_r = C_{n-p,n} =$
$C_{k+p+4,k+2p+4}$ (see Figure 10(a)). This completes the description of $f$ for this case. By using a similar argument as in Case 1 of Lemma 4.4, we can find that $\bigcup_{y \in N(x)} f(y) = \{1, 2\}$ for all vertices $x$ in $GP(n, k)$ with $f(x) = \emptyset$.

For the case where $\text{Pos}(\ell_2) < \text{Pos}(rc_1)$. In this case, only $Y_l$ and $Y_r$ have a different parity on the number of colored pairs with respect to those in Case 2 of Lemma 4.4. Thus only need to append one more colored pair $(0, 1) \to Y_l$ and set $Y_r = Y_l$. All the other colored pairs remain unchanged. Then, by using a similar argument as in Case 2 of Lemma 4.4, we can find that $\bigcup_{y \in N(x)} f(y) = \{1, 2\}$ for all vertices $x$ in $GP(n, k)$ with $f(x) = \emptyset$.

Now we consider the case where $\text{Pos}(\ell_2(A)) = \text{Pos}(rc_1(A))$. Namely, $3p + 4 = k - p$ which yields $k = 4p + 4$ and $n = 10p + 11$. To find a 2-RDF for those $GP(n, k)$, we use Pattern $C$ when $p \geq 2$. Accordingly, we have that $\text{Pos}(\ell_0(C)) = p + 3$, $\text{Pos}(\ell_1(C)) = p + 3 - k \equiv 7p + 10 \pmod{n}$, $\text{Pos}(\ell_2(C)) = 3p + 6$, $\text{Pos}(rc_0(C)) = 9p + 9$, $\text{Pos}(rc_1(C)) \equiv 3p + 2 \pmod{n}$, and $\text{Pos}(rc_2(C)) = 7p + 6$ (see Figure 10(c)). The coloring function $f$ for blocks $X_l, Y_l, Z_l, X_r, Y_r,$ and $Z_r$ is constructed as follows: Set $X_l = C_{p+5,3p+1} = (0^p \circ (10)_2)^{(p-2)}$, $Y_l = (10^1_{(020)})$, $Z_l = (2^3_6) \circ (02)^{p-2}_{(10)}$, $X_r = \overline{X}_l$, $Y_r = \overline{Y}_l$, $Z_r = \overline{Z}_l$, $P_l = C_{1,p} = C_{k+1,k+p}$, and $P_r = C_{n-p,n} = C_{k+p+4,k+2p+4}$ (see Figure 10(c)). By Propositions 2.1, 2.2, and 2.3, it is easy to check that $f$ is a 2-RDF of $GP(n, k)$ with $w(f) = n - 1$. For the case where $p = 1$, we have $(n, k) = (21, 8)$. We can find that $C_{1,21} = (20_{(012002100021020120021020120)})$ is a 2-RDF for $GP(21, 8)$ with $w(f) = 20$. This completes the proof. □
**Theorem 4.6.** For $n \geq 13$ and $n \neq 2k+2$, we have $\gamma_{r2}(\text{GP}(n,k)) \leq n-1$.

**Proof.** By Lemmas 3.3-4.5, the theorem follows directly. □

5. Concluding remarks

By Theorem 4.6, we disprove the following conjecture posed in [8]: $\gamma_{r2}(\text{GP}(n,k)) = n$ if and only if both $n < 4k+1$ and $n < 4k'+1$ hold, where $kk' \equiv \pm 1 \pmod{n}$ and $1 \leq k' \leq \lfloor (n-1)/2 \rfloor$.

As a future study, we pose the following conjecture: For $n \geq 12$, $\gamma_{r2}(\text{GP}(n,k)) = n$ if and only if $n = 2k + 2$.

**References**


